NOTES ON STATICS

AS ARRANGED FOR THE FIRST YEAR
IN THE
FACULTY OF
APPLIED SCIENCE AND ENGINEERING

BY
C. H. C. WRIGHT

UNIVERSITY OF TORONTO PRESS

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PREFACE

THESE notes have been written in the hope that they may be of some assistance to the returned men who are starting their first year in the Faculty of Applied Science and Engineering or who are resuming their studies in the higher years and wish to review the subject of Statics.

C. H. C. WRIGHT.

Engineering Building,
Jan. 2, 1919

NOTES ON STATICS

GRAPHICAL AND ANALYTICAL

Statics is that branch of Mechanics which treats of forces.

MECHANICS

Statics	Kinematics	Dynamics
Treats of Forces	Treats of Motion	Treats of Forces and Motion

Force is the action between two bodies, either causing or tending to cause change in their relative rest or motion.

Force is that which moves or tends to move or which changes or tends to change the motion of a body.

The notion of force is first obtained directly by sensation, for the forces exerted by the voluntary muscles can be felt. The existence of forces other than muscular tension is inferred from their effect.

Equilibrium is the condition of two or more forces which are so opposed that their combined action on a body produces no change in its rest or motion.

Force has magnitude, direction, sense and point of application.

Magnitude. Let the unit of magnitude be that force which is exerted by the earth on the standard pound of platinum kept in the Exchequer Office, London, England, when measured in the latitude of London at sea level.

Let the instrument for measuring forces be the standard spring. To calibrate this instrument, first, observe and mark the effect of the application of the pull of the standard pound. Then use some other weight which will produce the same effect; add the standard pound and mark the effect of the two-pound pull. Continue until the whole range of the spring is calibrated.

Direction. The direction of a force is the direction of the straight line along which the body tends to move in consequence of the force, *e.g.*, the horizontal or vertical directions—the direction of College Street, Yonge Street, etc.

Sense. The distinction between up and down in the vertical direction or between north and south in the direction of the meridian is called sense.

The *direction* of a force may be represented by the direction of a straight line and its sense by an arrow head placed on the line.

The *point of application* of a force is that point of the rigid body upon which the force acts. During this course of lectures the problems will be limited to those concerning forces acting in one plane only. The trusses will be considered as ideal, *i.e.*, those whose members are perfectly rigid without weight and intersect perfectly at the joints. The joints act without friction so that the only force exerted by any member must have the direction of the member.

Vector Polygon

Let AB and BC be two known forces acting on a point as indicated in Fig. 1.

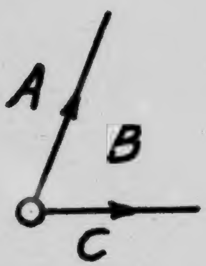


Fig 1.

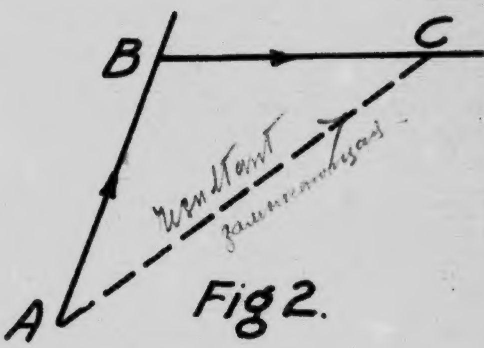


Fig 2.

From any point A (Fig. 2) draw the line AB parallel to the force AB and on some chosen scale cut off the length AB to represent the magnitude of the force AB. Place the arrow on the line to indicate the sense of the force. From B draw the line BC parallel to the force BC and similarly cut off the length BC to represent the magnitude of the force.

Thus AB and BC (Fig. 2) have been drawn to represent the known forces in such a manner that the sense marks are continuous from the initial point A to the final point C. The figure ABC is called a "Vector Polygon" and the line joining A and C will represent the resultant of the forces AB and BC; *i.e.*, the direction of AC will represent the direction of the resultant; the length of AC will represent the magnitude of the resultant and its sense will be from A towards C.

If AB and BC be two forces acting on a point as in Fig. 1, they may be removed and the single force represented by AC substituted as their equivalent.

The proof of this statement is experimental.

Let AB and BC (Fig. 3) be two strings fastened together at the point ABC and passing over pulleys D and E. Let the other ends of the strings be fastened to 10 lb. weights. Let there be applied a pull on the point ABC such that the strings assume the position indicated in Fig. 3; *i.e.*, make angles of 30° with the horizontal.

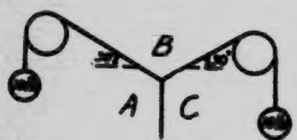


Fig 3.

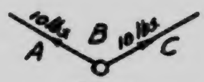


Fig 4

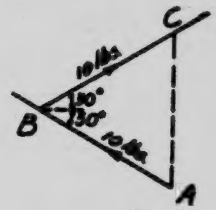


Fig 5

Now the point ABC is acted on by three forces—*i.e.* (1) the string AB will exert a pull on the point in the direction of the string of 10 lbs., (2) the string BC will exert a pull on the point in the direction of the string of 10 lbs., (3) the pull AC is unknown.

Consider first the known forces AB and BC as indicated in Fig. 4

From any point A (Fig. 5) draw the line AB parallel to the force AB and cut off a length AB to represent 10 lbs.

From B draw BC parallel to the force BC and cut off the length BC to represent 10 lbs. Then the figure ABC is a Vector Polygon and the line AC will represent the resultant.

Now the side AB is equal in length to the side BC and the angle ABC is 60°; therefore, the triangle ABC is an equilateral triangle and the angle BAC is equal to 60°.

But the side AB makes an angle of 30° with the horizontal, therefore the line AC is vertical.

Hence the resultant of AB and BC is a vertically upward force of 10 lbs., *i.e.*:

The forces AB and BC may be removed from the point ABC (Fig. 3) and a single vertical force with a magnitude of 10 lbs. and an upward sense substituted instead.

Therefore, the unknown force AC must be vertical, have a downward sense and a magnitude of 10 lbs.

Or, in other words, if a weight of 10 lbs. be suspended from the point ABC (Fig. 3) there will be equilibrium.

Try this experimentally and it will be found to be correct; hence, in this case the closing line of the Vector Polygon does represent the resultant.

Many similar experiments have been performed with the same result; therefore, the line which joins the initial to the final point of a Vector Polygon does represent the resultant in magnitude, direction and sense.

Let AB, BC and CD (Fig. 6) represent three known forces acting on a point where CD is a vertical pull of 8 lbs., BC a horizontal pull of 4 lbs. and AB a pull of 5 lbs. in the direction indicated in the sketch (Fig. 6).

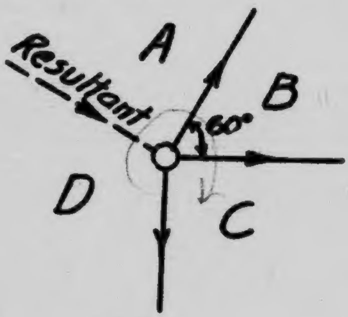


Fig 6.

Fig 7.

It is required to find their resultant.

From any point A (Fig. 7) draw AB parallel to the force AB and cut off a length AB to represent 5 lbs. Place the sense mark on the line. Similarly from B draw BC to represent the force BC and from C the line CD to represent the force CD.

Then $ABCD$ is a Vector Polygon, and AD must represent the resultant in magnitude, direction and sense. Its point of application is $ABCD$ (Fig. 6).

Let AB, BC, CD and DE (Fig. 8) represent four known forces as indicated. It is required to find their resultant.

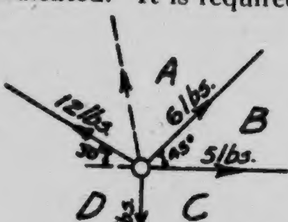


Fig 8.

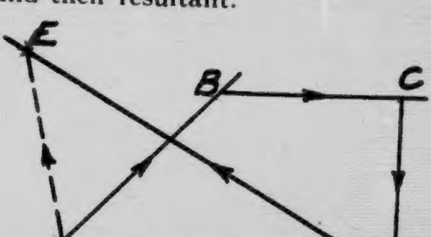


Fig. 9

From any point A (Fig. 9) draw the Vector Polygon $ABCDE$ as previously described. Then AE will represent the resultant.

Questions

Determine graphically the resultant of

1. Two horizontal forces 4 lbs. and 5 lbs. each with opposite senses.
2. Two horizontal forces of 4 lbs. and 5 lbs. each having the same senses.
3. A horizontal force of 5 lbs. with a sense towards the right and a vertical force of 5 lbs. with an upward sense.
4. A horizontal force of 10 lbs. towards the left and a vertical force of 10 lbs. with an upward sense.
5. A horizontal force of 5 lbs. towards the right, a force of 5 lbs. acting upward and towards the right at an angle of 60° to the horizontal and a horizontal force of $(8-3\sqrt{3})$ lbs. acting towards the left.

Stresses in the Members of a Cantilever

If a set of forces acting on a body are in equilibrium, their combined action does not tend to produce any change in its rest or motion, *i.e.*, their resultant is zero. Hence the closing line of the Vector Polygon in such a case will be without length, or the Vector Polygon is said to close. The final point must coincide with the initial point.

If any body such as AB (Fig. 10) is acted on by a single force P at A in its own direction and is kept in equilibrium by a second force P' at B, then P' must have the same direction and magnitude as P, but the opposite sense.

Fig 10.

Fig. 11.

Similarly if AB (Fig. 11) is pulled at A by Q, it must also be pulled on at B by Q' if equilibrium is to be maintained.

In the first case the member AB is said to be in a state of compression when every particle is pushing on the adjoining particles, while in the second case the member is said to be in a state of tension, *i.e.*, every particle is pulling on the adjoining particles.

Let the above figure (Fig. 12) represent a cantilever supporting a load of 1200 lbs. at the joint ABC.

Consider first the action of the forces on the pin ABC.

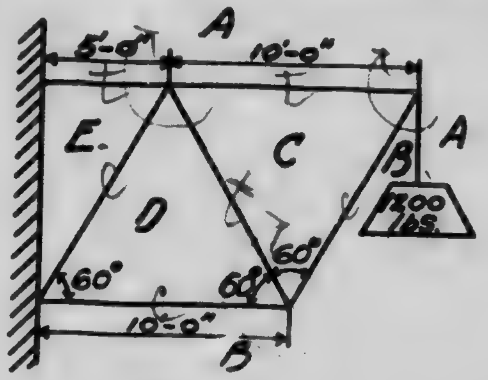


Fig. 12.

There are three members in contact with this pin, and hence there may be three forces acting on it, but no more (see Fig. 13).

Of these forces AB is known, while BC and CA are unknown except for their directions.

A B C A.

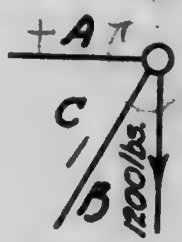


Fig. 13.

Fig. 14.

As these forces are in equilibrium, the Vector Polygon must close.

From any point A (Fig. 14) draw the line AB parallel to the force AB, cut off on a chosen scale the length AB to represent 1200 lbs. and place the sense mark on it.

From B draw a line parallel to BC and from A a line parallel to CA. Let these two lines intersect at the point C. Then $ABCA$ is the required Vector Polygon, and BC and CA will represent the two unknown forces in magnitude and sense.

R upper Comp. —
Of upper tens. +

Thus the member BC (Fig. 12) exerts a force on the pin that is represented by the line BC (Fig. 14), *i.e.*, it pushes on the pin and is therefore in a state of compression.

The magnitude of the push is determined by scaling the length of the line BC (Fig. 14). It is apparent that this is $800\sqrt{3}$ lbs.

Thus the compression in the member BC is $800\sqrt{3}$ lbs.

From the Vector Polygon (Fig. 14) it will be seen that the force CA acts towards the left, *i.e.*, the member CA (Fig. 12) tends to move the pin towards the left and is therefore pulling on it; hence, the member is in tension. The magnitude of the tension will be given by scaling the line CA (Fig. 14) or $400\sqrt{3}$ lbs.

Consider the forces acting on the pin BDC.

The member BC is in compression, therefore it pushes on the pin with a magnitude of $800\sqrt{3}$ lbs. as indicated in Fig. 15, while the forces BD and DC caused by the members BD and DC are unknown except for their directions.

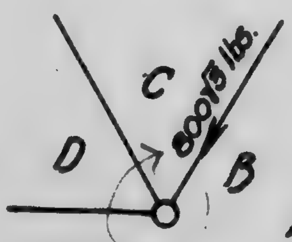


Fig. 15.

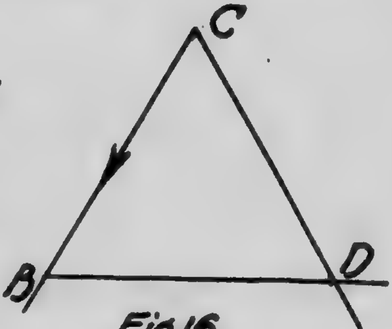


Fig. 16.

As these three forces are in equilibrium the Vector Polygon closes.

From any point C draw the line CB (Fig. 16) parallel to the force CB (Fig. 15) and cut off the length CB to represent the magnitude $800\sqrt{3}$ lbs.

From B draw BD parallel to the force BD and through C, CD parallel to the force CD; then $CBDC$ is the Vector Polygon and the force BD exerted by the member BD (Fig. 12) is towards the right; hence, the member pushes on the pin and is in a state of compression to the extent of $800\sqrt{3}$ lbs.

The force DC has an upward sense; therefore, the member pulls on the pin and is in a state of tension, the magnitude of which is $800\sqrt{3}$ lbs.

Consider the forces acting on the pin *ACDE*.

The member *AC* is in tension, therefore it pulls on the point with a pull of $400\sqrt{3}$ lbs.

The member *CD* is also in tension and pulls on the point with a pull of $800\sqrt{3}$ lbs.

The forces *DE* and *EA* exerted by the members *DE* and *EA* are unknown.

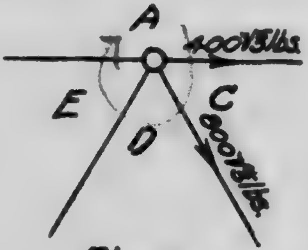


Fig. 17.

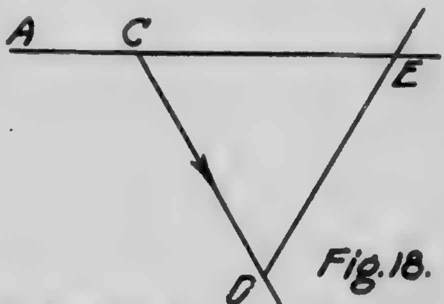


Fig. 18.

From any point *A* (Fig. 18) draw the line *AC* parallel to the force *AC* and mark off a length *AC* to represent $400\sqrt{3}$ lbs. From *C* draw *CD* to represent the force *CD*. Through *D* draw a line *DE* parallel to the direction of the unknown force *DE* and through *A* draw *AE* parallel to the force *AE* intersecting *DE* at the point *E*.

Then *ACDEA* is the Vector Polygon and *DE* and *EA* represent the two forces completely.

Hence the member *DE* is in compression to the extent of $800\sqrt{3}$ lbs. and *EA* is in tension, the magnitude of which is $1200\sqrt{3}$ lbs.

i.e., the stresses in the members of the cantilever have been determined.

BC	— compression	$-\ 800\sqrt{3}$ lbs.
CD	— tension	$+\ 800\sqrt{3}$ lbs.
DE	— compression	$-\ 800\sqrt{3}$ lbs.
DB	— compression	$-\ 800\sqrt{3}$ lbs.
CA	— tension	$+\ 400\sqrt{3}$ lbs.
EA	— tension	$+1200\sqrt{3}$ lbs.

It will be found convenient to use heavy lines in the truss diagram to represent members in *compression*, light lines those in *tension;* and in the Vector Polygon, to use heavy lines to represent those forces exerted by members in compression, while light lines represent those exerted by members in tension.

Place sense marks on the lines representing outside forces.

Drawing the Vector Polygons for the various points of the cantilever (Fig. 19) together will give Fig. 20.

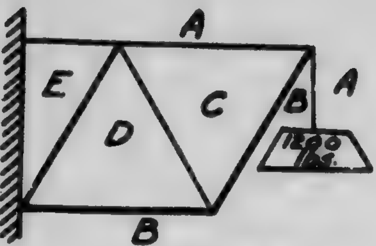


Fig. 19.

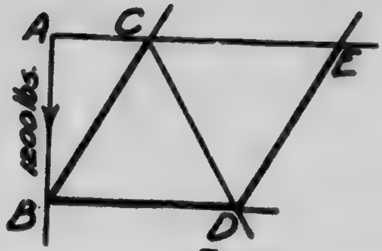


Fig. 20.

Determine the stresses in the members of a cantilever similar to Fig. 19, supporting three loads of 1200 lbs. each at the points *ABC*, *BEDC* and *ACDFG* (Fig. 21).

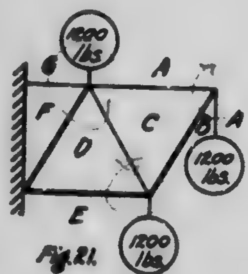


Fig. 21.

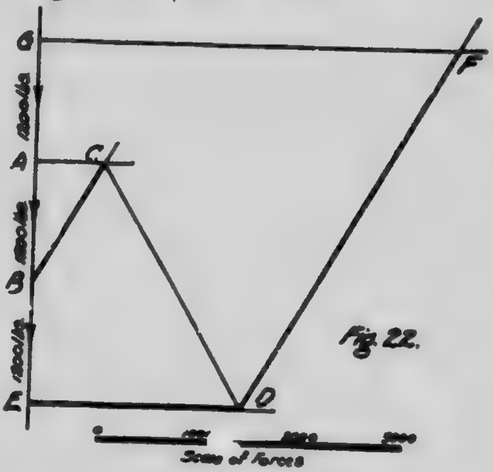


Fig. 22.

Determine the stresses in the cantilever designed and loaded as indicated in Fig. 23.

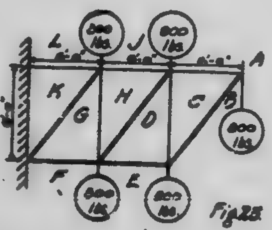


Fig. 23.

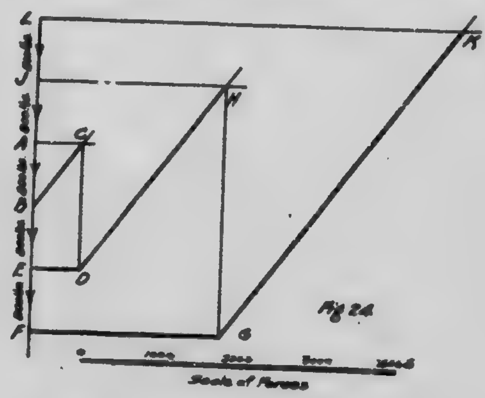
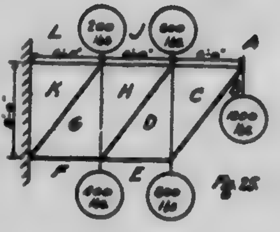
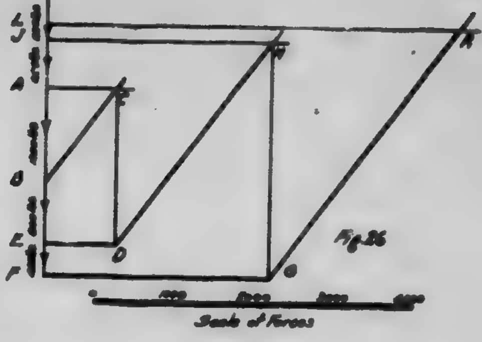


Fig. 24.





Determine the stresses in the cantilever designed and loaded as in Fig. 25.

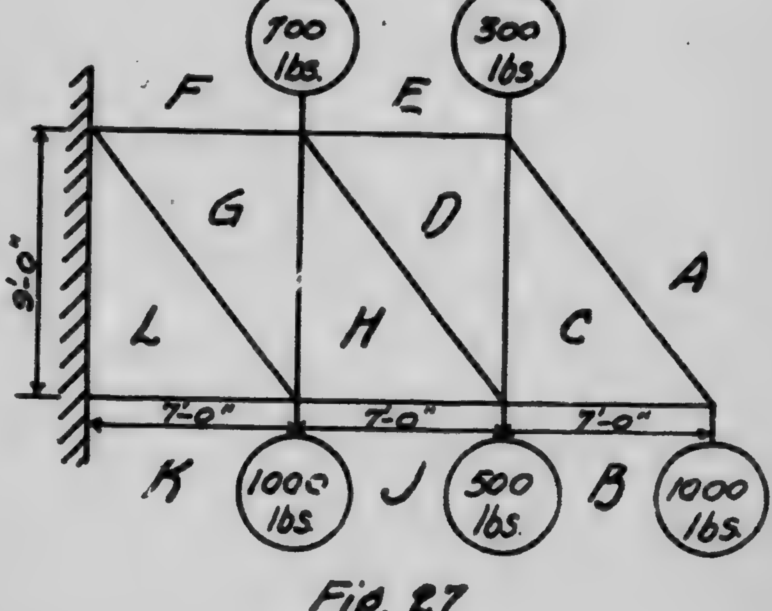


Fig. 27.

Determine the stresses in the above cantilever.

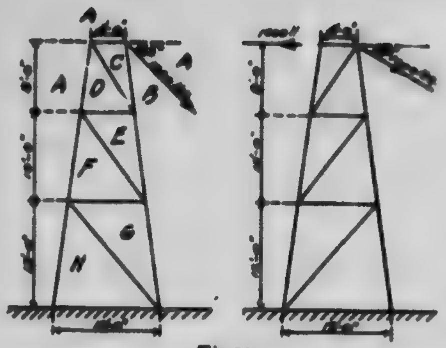


Fig. 28.

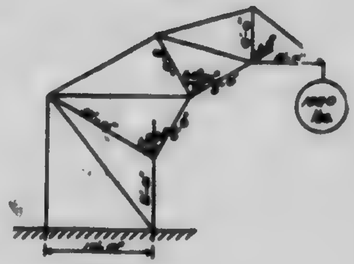


Fig. 29.

Determine the stresses in the above trusses (Figs. 28 and 29).

Resolved Parts

Let ABC (Fig. 30) be a Vector Polygon, then AC represents the resultant of the forces represented by AB and BC.

The forces AB and BC are called the components of AC.

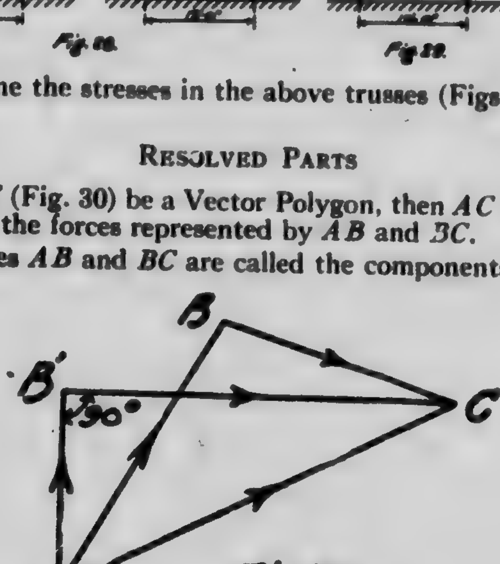


Fig. 30.

Similarly the forces represented by AB' and $B'C$ may be called a pair of components of AC.

Thus there may be an infinite number of pairs of components.

When the angle between the components is a right angle as at B' then the forces are not called components but resolved parts.

Thus AB' is the resolved part of AC in the direction AB' and is the efficiency of AC in the direction AB'.

The resolved part of any force such as AB (Fig. 31) in any given direction such as C may be determined by marking off along AB a length to represent the magnitude of AB, from A drawing a line parallel to the given direction C and through B dropping the perpendicular BD on it.

Then BD will represent the resolved part of AB in the given direction C. AD and DB represent two forces of which the resultant is represented by AB as ADB is a Vector Polygon, and the angle ADB is a right angle. Hence by definition, AD is the resolved part of AB in the direction AD or C.

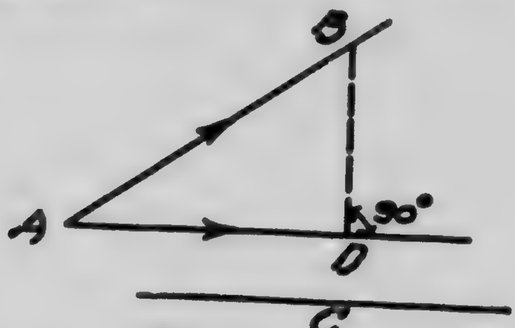


Fig. 31.

The resolved part of a force P in the horizontal direction is spoken of as the horizontal resolved part of the force P and written X_P. Similarly the vertical resolved part of the force P is written Y_P.

Let P and P' be two forces equal in magnitude acting as indicated in Fig. 32.

$$X_P = P \cos 30°.$$

$$X_{P'} = P' \cos 30°.$$

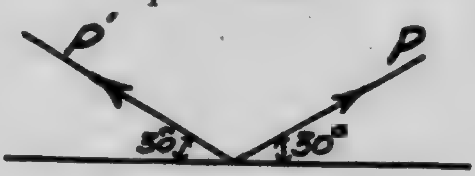


Fig. 32.

These horizontal resolved parts are equal in magnitude, $P \cos 30°$ acting towards the right and $P' \cos 30°$ acting towards the left. To distinguish between those different senses, it is customary to call one positive and the other negative. Generally that acting towards the right is assumed as $+ve$. Similarly vertical resolved parts with upward senses are assumed to be positive and those downward negative.

Let P and Q be two known forces acting on the point A (Fig. 33).

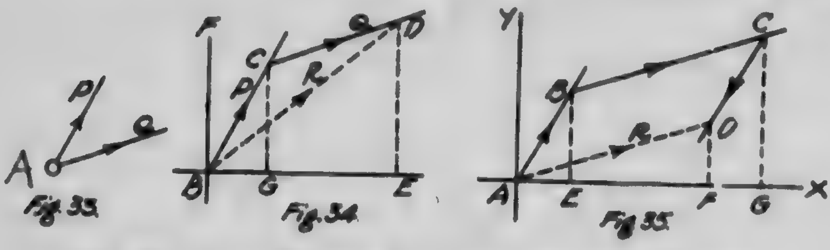


Fig. 33. Fig. 34. Fig. 35.

Draw the Vector Polygon BCD for these forces (Fig. 34).
Then BD will represent their resultant R.
Through B draw the horizontal and vertical lines BE and BF.
From C and D drop the perpendiculars CG and DE on BE.
Then

$$X_P = BG$$

$$\text{and } X_Q = GE$$

$$X_P + X_Q = BG + GE$$

$$= BE$$

$$= X_R.$$

i.e., the sum of the horizontal resolved parts of two forces is equal to the horizontal resolved part of their resultant. What is true for two forces must be true for any number of forces. Hence $\Sigma X = X_R$.
Similarly by dropping perpendiculars on BF it may be proved that $\Sigma Y = Y_R$.

Let $ABCD$ (Fig. 35) be a vector polygon, then AD represents the resultant R of the forces AB, BC and CD.

From B, C and D drop perpendiculars BE, CG and DF on AX

$$X_{AB} + X_{BC} + X_{CD} = = +AE + EG - FG$$

$$= AF$$

$$= XR$$

$$\therefore \Sigma X = XR$$

i.e., The algebraic sum of the horizontal resolved parts of any set of forces is equal to the horizontal resolved part of their resultant.

Similarly, it may be proved that the algebraic sum of the vertical resolved parts of any set of forces is equal to the vertical resolved part of their resultant,

$$i.e., \Sigma Y = Y_R.$$

Thus it has been proved that generally

(*a*) graphically

the closing line of the Vector Polygon represents the resultant.

(*b*) analytically

(1) $\Sigma X = X_R$

(2) $\Sigma Y = Y_R$

Again, if a set of forces is in equilibrium their resultant is 0. Hence in the special case of equilibrium

(*a*) Graphically

The Vector Polygon closes

(*b*) Analytically

(1) $\Sigma X = 0$

(2) $\Sigma Y = 0$

To determine analytically the stresses in the members of the cantilever (Fig. 36) supporting a load of 1200 lbs. at the outer end, consider first the three forces acting on the pin *ABC* (Fig. 37).

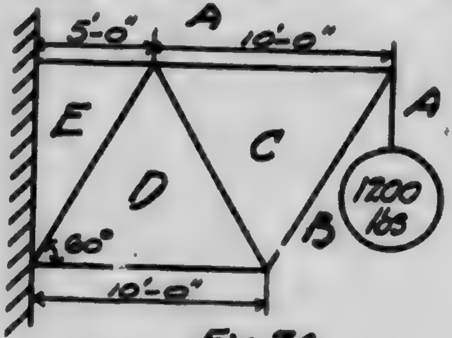


Fig. 36.

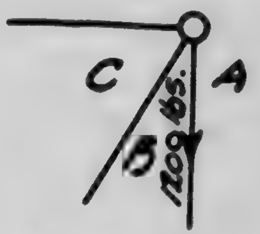


Fig. 37.

$$X_{AB} = 0$$

$$Y_{AB} = -1200$$

$$X_{BC} = BC \cos 60° = BC\tfrac{1}{2}$$

$$Y_{BC} = BC \sin 60° = BC\frac{\sqrt{3}}{2}$$

The sense of *BC* is unknown.

∴ the senses of its resolved parts are not known.

Assume that *BC* is a push, then its horizontal resolved part will be $+ve$ and its vertical resolved part $+ve$.

$$X_{CA} = CA$$

$$Y_{CA} = 0.$$

The sense of *CA* is unknown.

Assume that X_{CA} is $+ve$,

because these three forces are in equilibrium,

Then $\quad \Sigma X = 0,$

i.e. (1) $\quad X_{AB} + X_{BC} + X_{CA} = 0$

$$0 + \frac{BC}{2} + CA = 0$$

again (2) $\quad \Sigma Y = 0$

$$Y_{AB} + Y_{BC} + Y_{CA} = 0$$

$$-1200 + \frac{\sqrt{3}}{2} BC + 0 = 0$$

$$\therefore \quad BC = +\frac{1200 \times 2}{\sqrt{3}}$$

$$= +800\sqrt{3}$$

This positive sign means that the assumption that the force BC was a push is correct; therefore the member BC which makes this push is in compression.

From equation (1)

$$\frac{3C}{2} + CA = 0$$

$$400\sqrt{3} + CA = 0$$

$$CA = -400\sqrt{3}$$

This negative sign means that the assumption that the horizontal resolved part of CA was positive is not correct; therefore the force CA acts towards the left and the member CA must pull on the pin. Hence CA is in tension

BC = compression $800\sqrt{3}$ lbs.

CA = tension $\quad 400\sqrt{3}$ lbs.

Next consider the forces acting on the pin BDC (Fig. 38).

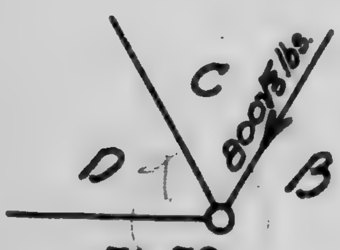


Fig. 38.

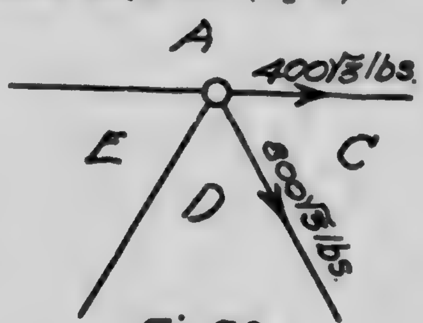


Fig. 39.

The member BC is in compression $\therefore$ the member pushes on the pin.

These three forces are in equilibrium; therefore (1) $\Sigma X=0$ and (2) $\Sigma Y=0$.

(1) $$X_{CB}+X_{BD}+X_{DC}=0$$

Assume that BD pushes and DC pulls on the pin

$$-400\sqrt{3}+BD-DC\cos 60^\circ=0$$

(2) $$Y_{CB}+Y_{BD}+Y_{DC}=0$$

$$-1200+0+DC\sin 60^\circ=0$$

$$DC\frac{\sqrt{3}}{2}=+1200 \text{ or } DC=+800\sqrt{3} \text{ lbs.}$$

This positive sign means that the assumption that the vertical resolved part of DC was positive is correct; hence the member DC pulls on the pin and is in tension.

Substituting in equation (1)

$$-400\sqrt{3}+BD-DC\cos 60^\circ=0$$

$$-400\sqrt{3}+BD-400\sqrt{3}=0$$

$$BD=+800\sqrt{3}$$

Hence the member BD pushes on the pin and is in compression.

Consider the forces acting on the pin $ACDE$ (Fig. 39)

$$X_{AC}=+400\sqrt{3} \text{ lbs.}$$

$$Y_{AC}=0$$

$$X_{CD}=+400\sqrt{3} \text{ lbs.}$$

$$Y_{CD}=-1200 \text{ lbs.}$$

Assume ED to be a push

then $$X_{ED}=+ED\cos 60^\circ$$

$$Y_{ED}=+ED\sin 60^\circ$$

Assume EA to be a push

then $$X_{EA}=+EA$$

$$Y_{EA}=0$$

As these four forces are in equilibrium, then $\Sigma Y=0$

$$Y_{AC}+Y_{CD}+Y_{DE}+Y_{EA}=0$$

$$0-1200+DE\sin 60^\circ+0=0$$

$$-1200+\frac{\sqrt{3}}{2}DE=0$$

$$DE=\frac{2}{\sqrt{3}}1200$$

$$=+800\sqrt{3}$$

The positive sign means that the member does push and is in compression.

$$\Sigma X = 0$$

$$X_{AC} + X_{CD} + X_{DE} + X_{EA} = 0$$

$$+400\sqrt{3} + 400\sqrt{3} + DE \cos 60° + EA = 0$$

$$+400\sqrt{3} + 400\sqrt{3} + 400\sqrt{3} + EA = 0$$

therefore $EA = -1200\sqrt{3}$

This negative sign means that the member EA does not push but pulls, and is in tension.

Hence

Members		*Stress*		*Amount*
BC	–	Compression	–	$800\sqrt{3}$ lbs.
CA	–	Tension	–	$400\sqrt{3}$ lbs.
DC	–	Tension	–	$800\sqrt{3}$ lbs.
BD	–	Compression	–	$800\sqrt{3}$ lbs.
DE	–	Compression	–	$800\sqrt{3}$ lbs.
EA	–	Tension		$-1200\sqrt{3}$ lbs.

Questions

By the method of resolved parts determine the stresses in the cantilevers illustrated in Figs. 21, 23, 25, and 27.

Law of Moments

Let P and Q (Fig. 40) be any two known forces and A any point distant a units from P and b from Q.

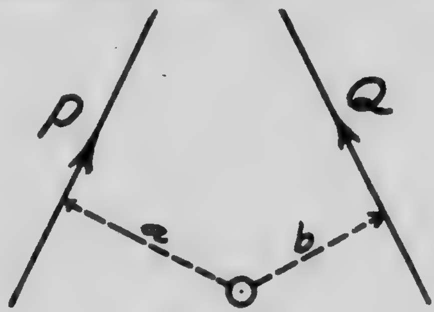


Fig. 40.

The product of P and a, i.e., $P.a$ is called the moment of the force P about the point A, and Qb the moment of Q about A. To distinguish between the sense of the moment of P which tends to turn clockwise from the sense of the moment of Q which is anti-clockwise it is customary to call one positive and the other negative,

usually the sense of turning with the hands of the clock is considered positive

$$\text{i.e.,}\quad M_P = +Pa$$
$$M_Q = -Qb$$

Let P and Q (Fig. 41) be any two forces and A any point; then

$$M_P = +Pa$$
$$M_Q = +Qb$$

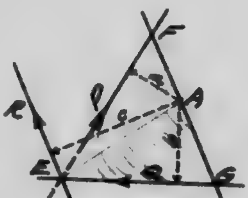


Fig. 41.

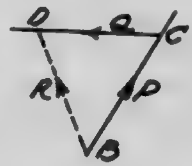


Fig. 42.

Fig. 43.

Draw the vector polygon BCD (Fig. 42). Then BD will represent in magnitude, direction and sense the resultant (R) of P and Q.

Produce the lines of directions of P and Q (Fig. 41) until they intersect at E. The resultant R must act through this point. Through E draw a line parallel to the direction of R and from A drop the perpendicular c on R. Then $M_R = +R{\cdot}c$.

Through A draw a line parallel to R intersecting P and Q at F and G respectively and join AE.

Because the triangle GEF has its sides parallel to Q, P and R, it may be considered as a vector polygon for these forces, thus GE may represent the force Q, $EF-P$, and $FG-R$.

Then $\quad M_P = +P{\cdot}a = +EF{\cdot}a$

$\quad M_Q = +Q{\cdot}b = +GE{\cdot}b$

and $\quad M_R = +R{\cdot}c = +GF{\cdot}c$

but $\quad EF{\cdot}a = 2$ area of triangle EFA

$\quad GE{\cdot}b = 2$ " " " GEA

and $\quad GF{\cdot}c = 2$ " " " GEF

therefore $M_P + M_Q$ may be represented by 2 area of the triangles EFA and GEA or 2 area of triangle GEF

but 2 area of triangle GEF represents M_R

therefore $\quad M_P + M_Q = M_R$

Hence the sum of the moments of any two forces about any point is equal to the moment of their resultant about the same point

$$i.e., \Sigma M = M_R$$

Suppose the point A were below the line of direction of Q as in Fig. 43, then

$$M_P = +Pa = 2 \text{ triangle } EFA$$

$$M_Q = -Qb = -2 \text{ triangle } GEA.$$

$$M_P + M_Q = 2 \text{ triangle } GFE = M_R$$

or

$$\Sigma M = M_R$$

Thus the general conditions for any set of forces are

$$(1) \ \Sigma X = X_R$$

$$(2) \ \Sigma Y = Y_R$$

$$(3) \ \Sigma M = M_R$$

The special conditions for a set of forces in a state of equilibrium are

$$(1) \ \Sigma X = 0$$
$$(2) \ \Sigma Y = 0$$
$$(3) \ \Sigma M = 0$$

Analytical Methods

Let the adjoining Fig. 44 represent a simple truss resting on two walls or abutments and carrying a load of 1,000 lbs. at the centre.

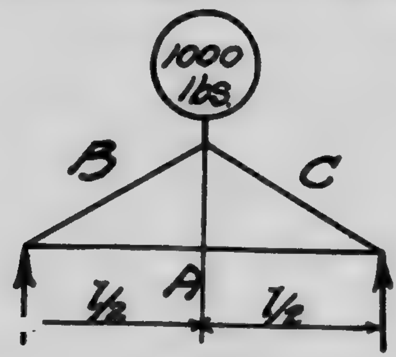


Fig. 44.

Consider the forces acting on the whole truss. They are AB, BC and CA.

As these three forces are in equilibrium, the laws of equilibrium must hold, viz.,

$$(1)\ \Sigma X = 0$$
$$(2)\ \Sigma Y = 0$$
and $$(3)\ \Sigma M = 0$$

From equation (3) we have

$$M_{AB} + M_{BC} + M_{CA}$$

about any point is equal to 0.

Take moments about any point in the line of direction of AC then

$$M_{AB} = +AB \cdot l$$

$$M_{BC} = -1000 \cdot \frac{l}{2}$$

$$M_{CA} = 0$$

$$\Sigma M = 0$$

$$+AB \cdot l - 1000 \cdot \frac{l}{2} + O = 0$$

$$AB = +\frac{1000 \times l}{l \times 2} = +500 \text{ lbs.}$$

again $$\Sigma Y = 0$$

therefore $$Y_{AB} + Y_{BC} + Y_{CA} = 0$$

$$+500 - 1000 + CA = 0$$

$$CA = 500 \text{ lbs.}$$

To determine analytically the stresses in a Warren Girder loaded as indicated in Fig. 45.

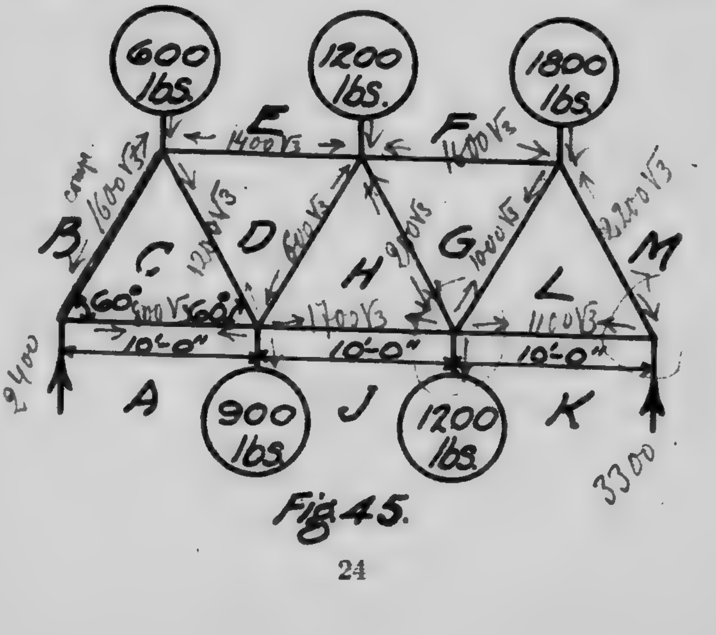


Fig. 45.

First consider the forces acting on the whole truss. They are AB, BE, EF, FM, MK, KJ and JA.

As these forces are in equilibrium the laws of equilibrium hold, viz., (1) $\Sigma X=0$, (2) $\Sigma Y=0$, and (3) $\Sigma M=0$.

Make the usual assumptions for the $+ve$ and $-ve$ senses for resolved parts and moments.

Take moments about a point in the line of direction of MK; then $M_{AB}+M_{BE}+M_{EF}+M_{FM}+M_{MK}+M_{KJ}+M_{JA}=0$

i.e., $+AB\times 30-600\times 25-1200\times 15-1800\times 5+0-1200\times 10$
$$-900\times 20=0$$

$$30AB-15{,}000-18{,}000-9{,}000-12{,}000-18{,}000=0$$
$$30\,AB=+72{,}000$$
$$AB=+2400 \text{ lbs.}$$

$\Sigma Y=0$

$\therefore \quad Y_{AB}+Y_{BE}+Y_{EF}+Y_{FM}+Y_{MK}+Y_{KJ}+Y_{JA}=0$

$+2400-600-1200-1800+MK-1200-900=0$

$\therefore \quad MK=+5700-2400=3300$ lbs.

Next consider the forces acting on the pin ABC. They are AB, BC and CA and are in equilibrium

$\therefore \quad$ (1) $\Sigma X=0$, (2) $\Sigma Y=0$, (3) $\Sigma M=0$

(2) $Y_{AB}+Y_{BC}+Y_{CA}=0$

$+2400+BC\sin 60^\circ+O=0$

$$BC\frac{\sqrt{3}}{2}=-2400$$
$$BC=-1600\sqrt{3} \text{ lbs.}$$

Hence the member BC is in compression to the extent of $1600\sqrt{3}$ lbs.

(2) $\Sigma X=0$

$X_{AB}+X_{BC}+X_{CA}=0$

$O-1600\sqrt{3}\cos 60^\circ+CA=0$

$$CA=+800\sqrt{3}$$

$\therefore \quad$ the tension in CA is $800\sqrt{3}$ lbs.

Consider the forces acting on the pin $CBED$

$\Sigma Y=0$

$Y_{CB}+Y_{BE}+Y_{ED}+Y_{DC}=0$

$+1600\sqrt{3}\sin 60^\circ-600+O+DC\sin 60^\circ=0$

$\times 2400-600+DC\frac{\sqrt{3}}{2}=0$

$$DC=-1200\sqrt{3}$$

$\therefore \quad$ the tension in DC is $1200\sqrt{3}$ lbs.

$$\Sigma X = 0$$
$$X_{CB} + X_{BE} + X_{ED} + X_{DC} = 0$$
$$1600\sqrt{3} \times \cos 60° + O + ED + 1200\sqrt{3} \cos 60° = 0$$
$$ED = -1400\sqrt{3}$$

$\therefore$ the compression in ED is $1400\sqrt{3}$ lbs.

Consider the forces acting on the pin $JACDH$

$$\Sigma Y = 0$$
$$Y_{JA} + Y_{AC} + Y_{CD} + Y_{DH} + Y_{HJ} = 0$$
$$-900 + O + 1200\sqrt{3} \sin 60° + DH \sin 60° + O = 0$$
$$-900 + 1800 + DH \frac{\sqrt{3}}{2} = 0$$
$$DH = -\frac{900 \times 2}{\sqrt{3}} = -600\sqrt{3}$$

$\therefore$ the compression in DH is $600\sqrt{3}$ lbs.

$$\Sigma X = 0$$
$$X_{JA} + X_{AC} + X_{CD} + X_{DH} + X_{HJ} = 0$$
$$O - 800\sqrt{3} - 1200\sqrt{3} \cos 60° - 600\sqrt{3} \cos 60° + HJ = 0$$
$$-800\sqrt{3} - 600\sqrt{3} - 300\sqrt{3} + HJ = 0$$
$$HJ = +1700\sqrt{3}$$

$\therefore$ the tension in HJ is $1700\sqrt{3}$ lbs.

Consider the forces acting on the pin $HDEFG$

$$\Sigma Y = 0$$
$$Y_{HD} + Y_{DE} + Y_{EF} + Y_{FG} + Y_{GH} = 0$$
$$+600\sqrt{3} \sin 60° + O - 1200 + O + GH \sin 60° = 0$$
$$+900 - 1200 + GH \frac{\sqrt{3}}{2} = 0$$
$$GH = +\frac{300 \times 2}{\sqrt{3}} = +200\sqrt{3}$$

$\therefore$ the compression in GH is $200\sqrt{3}$ lbs.

$$\Sigma X = 0$$
$$X_{HD} + X_{DE} + X_{EF} + X_{FG} + X_{GH} = 0$$
$$600\sqrt{3} \cos 60° + 1400\sqrt{3} + O + FG - 200\sqrt{3} \cos 60° = 0$$
$$+300\sqrt{3} + 1400\sqrt{3} + FG - 100\sqrt{3} = 0$$
$$FG = -1600\sqrt{3}$$

$\therefore$ the compression in FG is $1600\sqrt{3}$ lbs.

Consider the forces acting on the pin $KJHGL$

$$\Sigma Y = 0$$
$$Y_{KJ} + Y_{JH} + Y_{GH} + Y_{GL} + Y_{LK} = 0$$

$$-1200+O-200\sqrt{3}\sin 60^\circ+GL\sin 60^\circ+O=0$$

$$-1200-300+GL\frac{\sqrt{3}}{2}=0$$

$$GL=+\frac{1500\times 2}{\sqrt{3}}=+1000\sqrt{3}$$

$\therefore$ the tension in GL is $1000\sqrt{3}$ lbs.

$\Sigma X=0$

$$X_{KJ}+X_{JH}+X_{HG}+X_{GL}+X_{LK}=0$$

$$0-1700\sqrt{3}+200\sqrt{3}\cos 60^\circ+1000\sqrt{3}\cos 60^\circ+LK=0$$

$$-1700\sqrt{3}+100\sqrt{3}+500\sqrt{3}+LK=0$$

$$LK=+1100\sqrt{3}\text{ lbs.}$$

$\therefore$ the tension in the member LK is $1100\sqrt{3}$ lbs.

Consider the forces acting on the point $LGFM$

$\Sigma Y=0$

$$Y_{LG}+Y_{GF}+Y_{FM}+Y_{ML}=0$$

$$-1000\sqrt{3}\sin 60^\circ+0-1800+ML\sin 60^\circ=0$$

$$-1500-1800+ML\frac{\sqrt{3}}{2}=0$$

$$ML=+\frac{3300\times 2}{\sqrt{3}}=+2200\sqrt{3}$$

$\therefore$ the compression in the member ML is $2200\sqrt{3}$ lbs.

Consider the forces acting on the point KLM

$\Sigma Y=0$

$$Y_{KL}+Y_{LM}+Y_{MK}=0$$

$$0-2200\sqrt{3}\sin 60^\circ+3300=0$$

$-3300+3300=0$ which is true

$\Sigma X=0$

$$X_{KL}+X_{LM}+X_{MK}=0$$

$$-1100\sqrt{3}+2200\sqrt{3}\cos 60^\circ+0=0$$

$-1100\sqrt{3}+1100\sqrt{3}=0$ which is true.

Members	*Condition of Stresses*	*Value of Stress*
BC	Compression	$1600\sqrt{3}$ lbs.
CD	Tension	$1200\sqrt{3}$ "
DH	Compression	$600\sqrt{3}$ "
HG	Compression	$200\sqrt{3}$ "
GL	Tension	$1000\sqrt{3}$ "
LM	Compression	$2200\sqrt{3}$ "
FD	Compression	$1400\sqrt{3}$ "
FG	Compression	$1600\sqrt{3}$ "
CA	Tension	$800\sqrt{3}$ "
HJ	Tension	$1700\sqrt{3}$ "
LK	Tension	$1100\sqrt{3}$ "

Examples

Determine analytically the stresses in the Howe Truss loaded as in Fig. 46.

Determine analytically the stresses in the Pratt Truss loaded as in Fig. 47.

Determine analytically the stresses in the Warren Girder loaded as in Fig. 48.

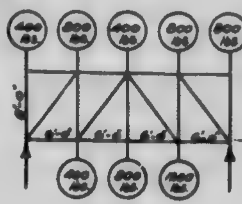


Fig. 46.

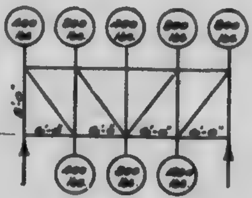


Fig. 47.

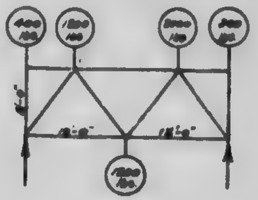


Fig. 48.

Method of Sections

To find the stress in the member NP of the Howe Truss represented in Fig. 49 supporting loads of 600 lbs. each at the joints of the upper chord and of 1200 lbs. each at the joints of the lower chord.

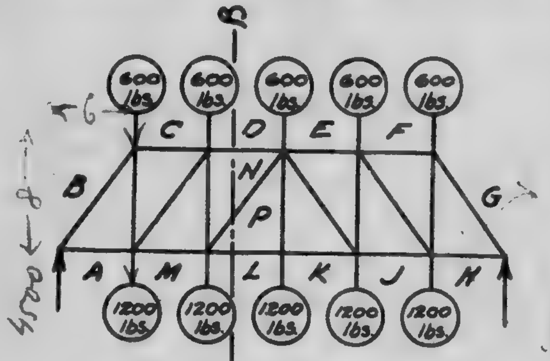


Fig. 49.

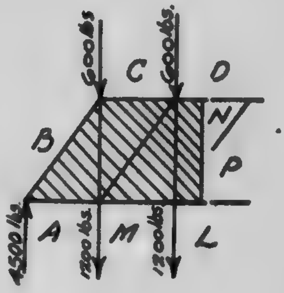


Fig. 50.

First consider the whole truss as a rigid body. The forces acting on it are in equilibrium; therefore $\Sigma M = 0$. Take moments about any point in the line of action of the force GH.

$$+AB \times 6 \times 6 - 1800 \times 5 \times 6 - 1800 \times 4 \times 6 - 1800 \times 3 \times 6 - 1800 \times 2 \times 6 - 1800 \times 1 \times 6 = 0$$

$$AB \times 6 \times 6 = +1800 \times 6\,(5+4+3+2+1)$$

$$AB = \frac{1800 \times 15}{6} = 4500 \text{ lbs.}$$

Next consider the portion of the truss to the left of the plane $\alpha\beta$ as a rigid body.

The forces acting on it are indicated in Fig. 50. As this portion of the truss is in equilibrium the forces acting on it are also in equilibrium

$$\therefore \quad (1)\ \Sigma X=0$$
$$(2)\ \Sigma Y=0$$
$$(3)\ \Sigma M=0$$

$$Y_{AB}+Y_{BC}+Y_{CD}+Y_{DN}+Y_{NP}+Y_{PL}+Y_{LM}+Y_{MA}=0$$

$$+4500-600-600+0+NP\times\frac{4}{5}+0-1200-1200=0$$

$$\frac{4}{5}NP=-\ 900$$

$$NP=-1125$$

The negative sign means that the force NP exerted by the right hand part of the member NP on the left is not a pull but a push; therefore, the compression in the member NP is 1125 lbs.

To find the stress in the member LP produce the directions of the forces PN and ND (Fig. 50) until they intersect, and take moments about the intersection.

$$M_{AB}+M_{BC}+M_{CD}+M_{DN}+M_{NP}+M_{PL}+M_{LM}+M_{MA}=0$$

$$+4500\times3\times6-600\times2\times6-600\times1\times6+0+0+LP\times8$$
$$-1200\times1\times6-1200\times2\times6=0$$

$$4500\times3-600\times2-600\times1+\frac{LP\times8}{6}-1200\times1-1200\times2=0$$

$$13500-5400+\frac{LP\times8}{6}=0$$

$$LP=-\frac{8100\times6}{8}=-6075 \text{ lbs.}$$

This negative sign means that the moment of the force LP about the point is negative; hence, the force LP is a pull and the member LP is in tension.

Questions

Use the method of sections to determine the stresses in two or more members of similarly loaded Pratt and Warren Girders; also use this method to check over the calculations made in previous exercises.

Determine by method of sections the stress in the main horizontal tie of a Fink Roof Truss supporting loads of 1000 lbs. at each joint. The span of the truss is 80 ft. and the height 20 feet.

Let the annexed diagram, Fig. 51, represent a Fink roof truss supporting the loads AB, BC, CD, DE, etc., and let the reaction of the left wall be MA.

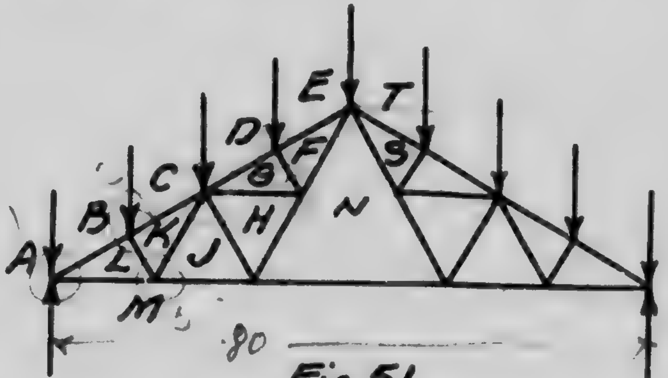


Fig. 51.

Consider first the forces acting on the point $ABLM$. There are two known forces MA and AB and two unknown—BL and ML exerted by the members BL and ML on the point as in Fig. 52.

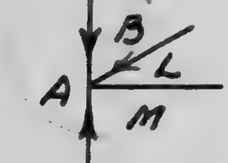


Fig. 52.

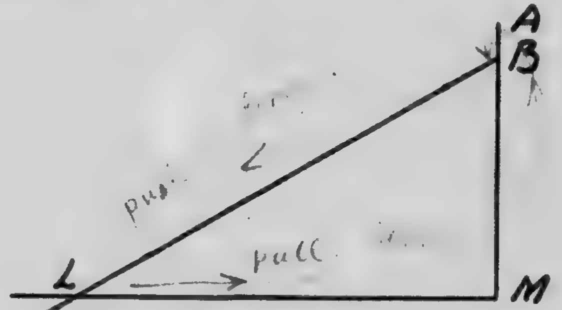


Fig. 53.

From any point M, Fig. 53, draw the lines MA and AB to represent the wall reaction MA and the load AB. Through B and M draw the line BL and ML parallel to the directions of the forces BL and ML. Let these lines intersect at L. Then $MABLM$ is the vector diagram for the point, and the lengths of BL and LM represent the magnitudes of the forces BL and LM acting on the point. The force BL being a push and LM a pull, hence the member BL is in compression and LM in tension.

Proceeding to the point $BCKL$ the known forces acting are LB and BC and the unknown CK and KL as in Fig. 54.

From any point L, Fig. 55, draw the line LB parallel to the force LB and from it cut off the length LB to represent the magnitude of the force, and from B draw BC to represent the force BC.

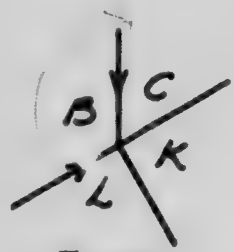


Fig. 54.

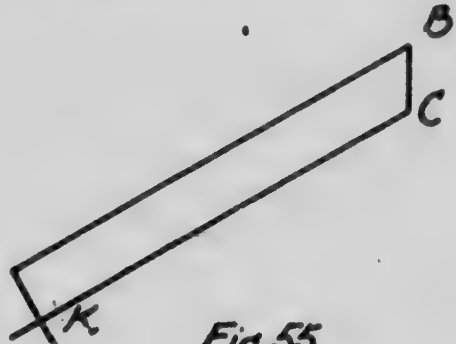


Fig. 55.

Through C draw CK parallel to the force CK and through L draw LK parallel to the force LK intersecting CK in the point K.

Then $LBCKL$ is the vector diagram for the point, and CK and KL represent the forces CK and KL. These are both pushes on the point, and therefore the members CK and KL are both in compression.

Considering the forces acting on the point $JKLM$ there are two known forces ML and LK and two unknown KJ and JM as in Fig. 56.

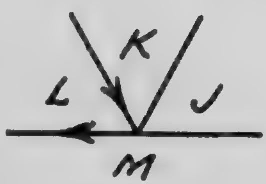


Fig. 56.

Fig. 57.

The vector diagram being $MLKGM$, Fig. 57, where KJ and JM represent the forces KJ and JM. As they are both pulls on the point the members KJ and JM are in tension.

Now examine the conditions existing at the point *DEFG*. There is one known force *DE* and three unknown, *EF*, *EG* and *GD*, as indicated in Fig. 58.

Two of these forces *DG* and *EF* act in the same direction and will have a resultant acting in this same direction. Substitute for these two forces their resultant *R*, making the set acting on the point *DE*, *GF* and *R*, Fig. 59.

Draw the vector diagram, Fig. 60, for these three forces. The lines *DE*, *GF* and *R* will represent the forces *DE*, *GF* and *R*, and as *GF* is a push on the point the member *GF* is in compression.

Fig. 58.

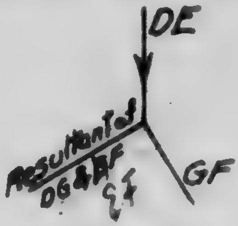


Fig. 59.

Fig. 60.

At the point *FGHN* there are four forces acting, one of which *FG* is known and the others *FN*, *NH* and *HG* are unknown and act as in Fig. 61.

Of the unknown forces *FN* and *NH* act in the same direction and will have a resultant acting in that direction. Substituting this resultant for the two forces the set of forces becomes *GF*, *GH*, R_2, Resultant of *FN* and *NH* (Fig. 62).

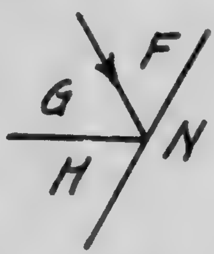


Fig. 61.

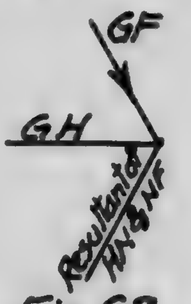


Fig. 62.

Fig. 63.

Draw the vector diagram *GF*, *GH*, R_2, Fig. 63, and the length of the line *GH* gives the magnitude of the tension in the member *GH*.

Combine these four vector diagrams in one, Fig. 64.

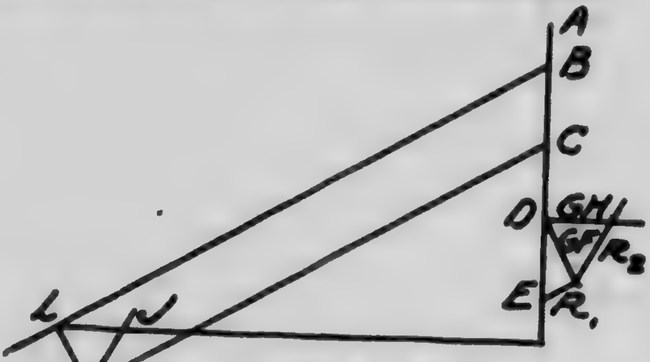


Fig. 64.

The diagram, Fig. 65, represents the condition existing at the point *CDGHJK*. There are two unknown forces *DG* and *HJ*.

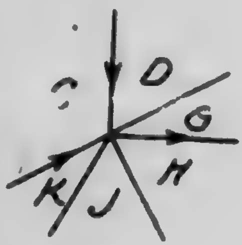


Fig. 65.

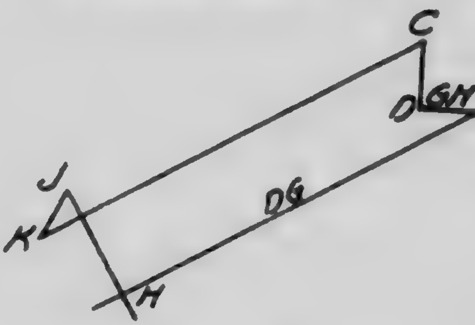


Fig. 66.

Draw the vector diagram *JKCD, GH, DG, HJ*, Fig. 66, and the length of *DG* and *HJ* will give the magnitude of the unknown forces.

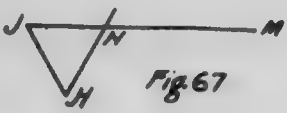


Fig. 67

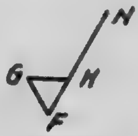


Fig. 68.

Fig. 69.

The vector diagrams for the points *MJHN*, *NHGF* and *DEFG* are given in Figs. 67, 68 and 69 respectively.

Adding these four vector diagrams to Fig. 64, completes the combined diagram as in Fig. 70.

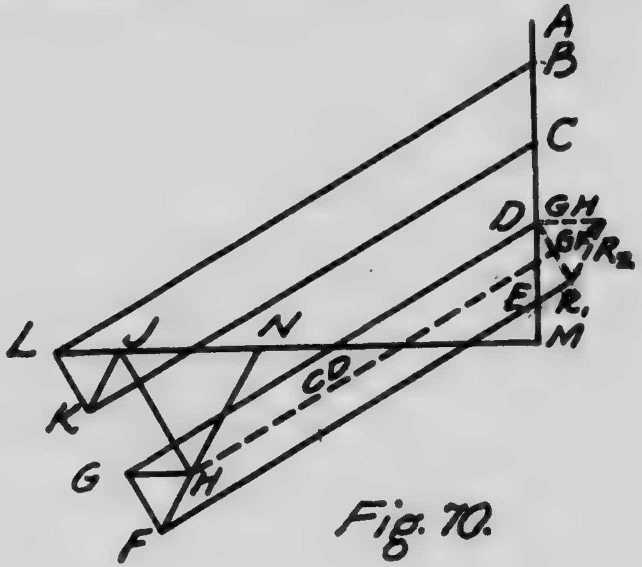


Fig. 70.

Suppose the loads *AB*, *BC*, *CD*, *DE* and *EF* are unequal, that their total is equal to the load on the right hand principle and that

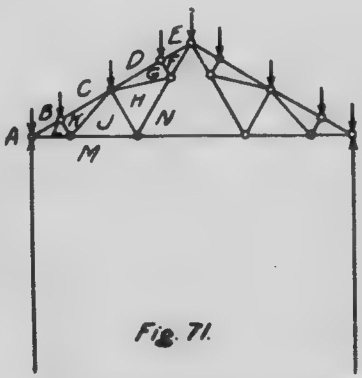


Fig. 71.

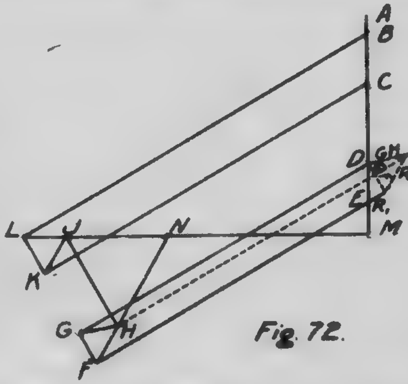


Fig. 72.

the lengths of the members *BL*, *CK*, *DG* and *EF* are unequal as in Fig. 71. Proceed as in the above problem and construct the vector diagram, Fig. 72.

Questions

Determine the stresses in the members of a French roof truss. It is customary to give the lower chord of the Fink truss a camber to improve its appearance.

Suppose the member *NA*, Fig. 73, is 1 foot above the horizontal line joining the ends of the principal rafters, determine the stresses in the members.

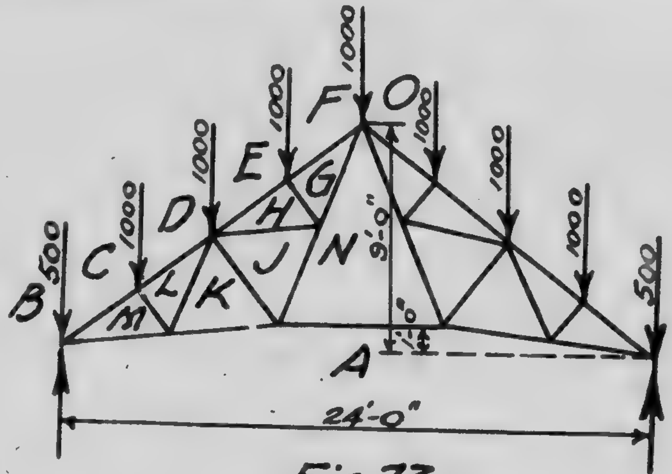


Fig. 73.

The Funicular Polygon

To determine graphically the position of the resultant of a set of forces acting on a rigid body.

Let *AB*, *BC* and *CD* be any three forces acting on a rigid body as indicated in Fig. 74.

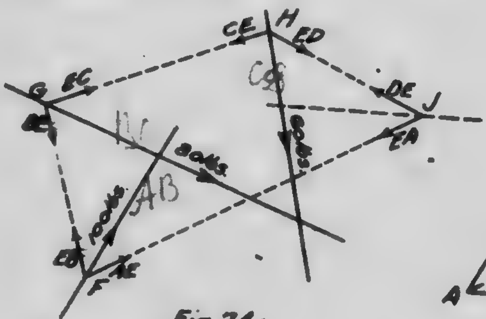


Fig. 74.

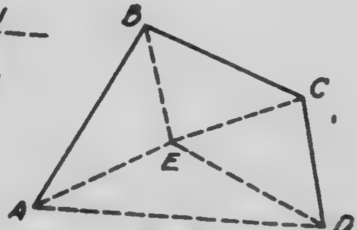


Fig. 75.

Draw the Vector Polygon *ABCD*, Fig. 75. Then *AD* will represent the resultant in magnitude, direction and sense.

Select any point E and join E with A, B, C and D. At any point F, Fig. 74, replace the force AB by a pair of components represented by AE and EB. Produce EB until it intersects the direction of the force BC at G, and at G replace the force by its
BE = EB components BE and EC. At H where EC intersects CD replace CD by components CE and ED. Then the original forces have been replaced by AE and EB acting at F, BE and EC acting at G and CE and ED at H. Of these six forces EB acting at F and BE at G are equal in magnitude, opposite in sense, and act in the same straight line; therefore their resultant is 0. Similarly EC and CE act in the same straight line with equal magnitude and opposite senses.

Thus the original forces AB, BC and CD may be replaced by ED acting at H and AE at F.

Produce these directions until they intersect at J. At J replace AE and ED by their resultant AD, Fig. 75.

Thus the resultant of AB, BC and CD is AD and acts through the point J.

The figure $FGHJ$ is called a funicular polygon.

SUMMARY

GENERAL CONDITIONS FOR ANY SET OF FORCES.

Graphical (*a*) The *Vector Polygon* gives the magnitude, direction and sense of the resultant.

(*b*) The *Funicular Polygon* gives the position of the resultant.

Analytical (*a*) (1) $\Sigma X = X_R$

(2) $\Sigma Y = Y_R$

(*b*) $\Sigma M = M_R$.

CONDITIONS FOR A SET OF FORCES IN EQUILIBRIUM.

Graphical (*a*) The Vector Polygon closes.

(*b*) The Funicular Polygon closes.

Analytical (*a*) (1) $\Sigma X = 0$

(2) $\Sigma Y = 0$

(*b*) $\Sigma M = 0$.

Graphical Statement	*Corresponding Analytical Statement*	
Closing line of Vector Polygon gives R	$\Sigma X = X_R$, $\Sigma Y = Y_R$	$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$
Intersection of final lines of Funicular Polygon gives position of R	$\Sigma M = M_R$	hence position of R

Couple

Let P and P_1 be two forces whose magnitudes are equal, directions parallel and senses opposite, and act as indicated in Fig. 76.

Draw the Vector Polygon AB, BA, Fig. 77. Select any point C and join CA and CB.

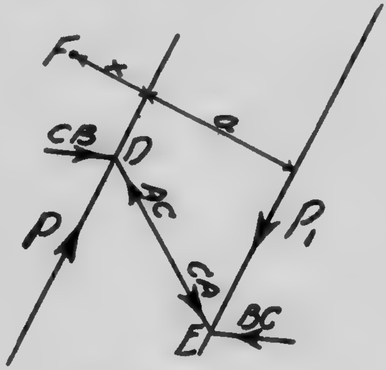


Fig. 76.

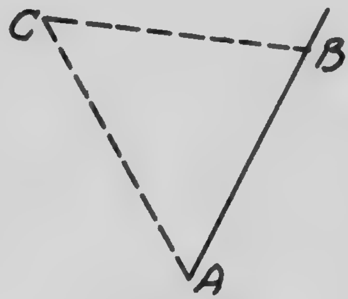


Fig. 77.

At any point D in the line of direction of P replace P by components AC and CB. Produce AC to intersect P_1 at E and at E replace P_1 by BC and CA.

P and P_1 are thus equivalent to CB and [illegible]ng at D and CA and BC at E. Of these four forces AC and [illegible] with equal magnitude and opposite senses along the same st[illegible]ght line, so that their resultant is zero.

Thus P and P_1 are equivalent to CB acting at D and BC at E.

The Vector Polygon AB, BA closing gives the appearance of equilibrium; but, the Funicular Polygon shows that a pair of parallel forces can be replaced only by another pair of parallel forces, or can be kept in equilibrium only by a second pair of parallel forces.

Such a pair of parallel forces as P and P_1 is called a couple.

Thus when the Vector Polygon closes and the Funicular Polygon remains open, it is proof that the set of forces is equivalent to a couple and equilibrium can only be maintained by introducing a couple with the opposite turning effect.

Analytical Analysis

$$X_P = +P \cos a \qquad Y_P = +P \sin a$$

$$X_{P_1} = -P_1 \cos a \qquad Y_{P_1} = -P_1 \sin a$$

$$X_P + X_{P_1} = 0 \qquad Y_P + Y_{P_1} = 0$$

Take moments about any point F distant x from P.

Then $\Sigma M = M_P + M_{P_1} = -P\cdot x + P_1(x+a) = +Pa$, *i.e.*, the algebraic sum of the moments of a couple about any point is constant and equal to the product of one of the forces and the distance between the forces.

Analytical Conditions of a Couple

$$(1)\ \Sigma X = 0$$
$$(2)\ \Sigma Y = 0$$
$$(3)\ \Sigma M = C$$

The conditions for equilibrium are

$$(1)\ \Sigma X = 0$$
$$(2)\ \Sigma Y = 0$$
$$(3)\ \Sigma M = 0$$

Therefore a couple can be balanced by a second couple only, the conditions of the balancing couple being

$$(1)\ \Sigma X = 0$$
$$(2)\ \Sigma Y = 0$$
$$(3)\ \Sigma M = -C$$

Examples

Determine the resultant of three forces of 10 lbs. each acting continuously around the sides of an equilateral triangle whose sides are 10 feet long.

Beam

Let the adjoining Fig. 78 represent a simple horizontal beam resting on two supports A and B and carrying a load of W lbs. at its centre.

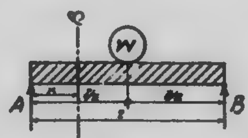


Fig. 78.

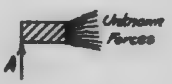


Fig. 79.

Fig. 80

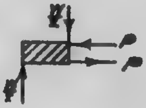


Fig. 81

Consider the beam as a rigid body. The forces acting on it are the two abutment reactions A and B and the load W. Take moments about B. Then

$$M_A + M_W + M_B = 0$$

$$+A1 - W\tfrac{1}{2} + 0 = 0$$

$$A = \frac{W}{2}$$

Similarly $$B = \frac{W}{2}$$

Let $\alpha\beta$ be any plane distant x from A.

The forces acting on the section of the beam to the left of $\alpha\beta$ are the abutment reaction A and the action of the right hand portions of the different fibres of the beam on the left hand portions which may be represented by Fig 79.

These unknown forces may each be replaced by its horizontal and vertical resolved parts as in Fig. 80.

As the original forces were in equilibrium, this set must also be in equilibrium.

$$\therefore \quad \Sigma X = 0, \ \Sigma Y = 0 \text{ and } \Sigma M = 0.$$

Hence the sum of the vertical resolved parts must be $-\frac{W}{2}$ and thus form with A a couple whose moment is $+\frac{W}{2}x$. Therefore the horizontal resolved parts must form a couple whose moment is $-\frac{W}{2}$ as suggested in Fig. 81.

Let the adjoining Fig. 82 represent a beam supporting loads of 400 lbs., 800 lbs. and 1200 lbs. as indicated.

By taking moments, the abutment reactions may be determined as

$$AB = 1{,}000 \text{ lbs.}$$

$$EA = 1{,}400 \text{ lbs.}$$

For Vertical loading the *Vertical Shearing Force* at any plane $\alpha\beta$ is the algebraic sum of the forces acting on the beam to the left of the plane, *i.e.*, when x is less than 5′ V.S.F. $= +1000$

when x is > 5 and < 10 V.S.F. $= +1000 - 400 = +600$

when x is > 10 and < 15 V.S.F. $= +1000 - 400 - 800 = -200$

when x is > 15 and < 20 V.S.F. $= +1000 - 400 - 800 - 1200 = -1400$

i.e., as x varies from 0 to 20′ the V.S.F. changes as the ordinates to the line $FGHJ-N$, Fig. 83.

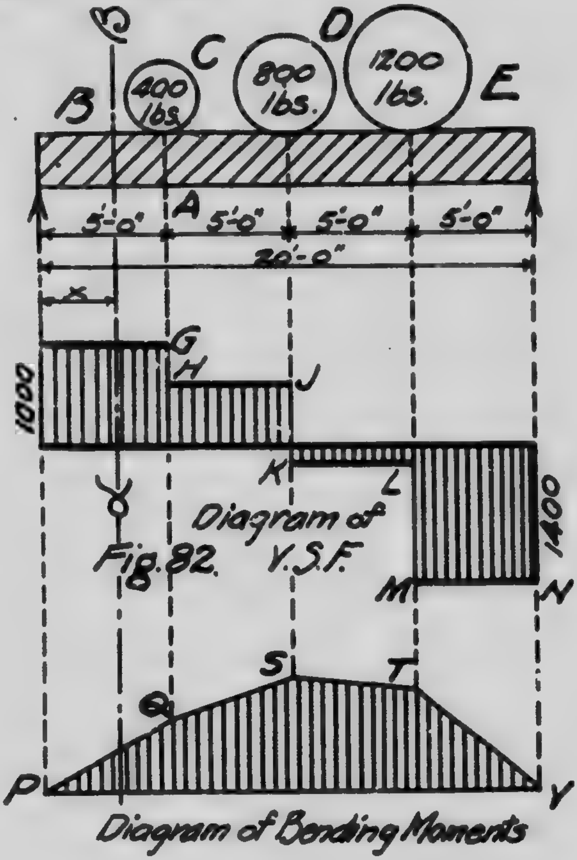


Fig. 83.

The *Bending Moment* at any plane $\alpha\beta$ is the algebraic sum of the moments of all the forces acting on the part of the beam to the left of $\alpha\beta$ about any point in the plane.

When $x<5'$ B.M. $=1000xx$

Let y represent B.M.

Then $y=1000\cdot x$

i.e., equation to the straight line PQ, Fig. 83.

Similarly when

$$x>5'<10' \text{ B.M.}=1000\times x-400(x-5).$$

Hence B.M. may be represented by the ordinates to the straight line QS.

When $x > 10$ and < 15, B.M. $= 1000 \times x - 400(x-5) - 800(x-10)$ and is represented by ordinates to line ST.

When $x > 15$ and < 20

$$\text{B.M.} = 1000 \times x - 400(x-5) - 800(x-10) - 1200(x-15).$$

When $x = 0$ B.M. $= 0$.

Thus the B.M. at any plane $\alpha\beta$ is represented by the diagram $PQSTV$, Fig. 83.

Let the adjoining Fig. 84 represent a simple horizontal beam supporting a load of W lbs. uniformly distributed over its length.

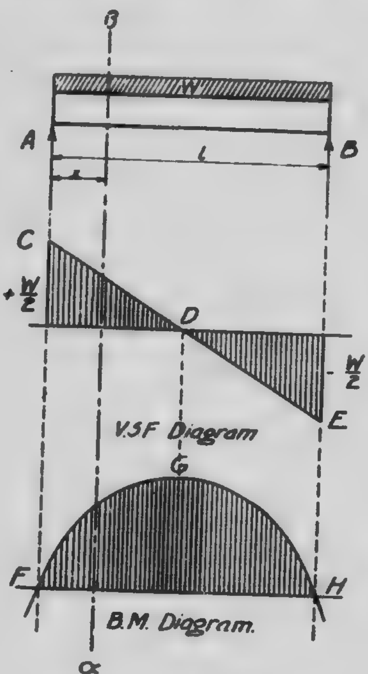


Fig. 84.

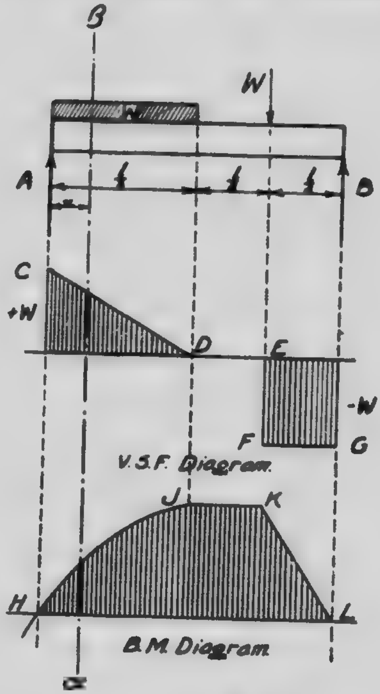


Fig. 85.

The abutment reactions A and B will each be $\frac{W}{2}$ lbs.

The V.S.F. at any plane $\alpha\beta = +A - \frac{W}{l} x$

Let y represent V.S.F.

then $y = A - \frac{W}{l} x$

when $x = 0$ $y = +A = +\frac{W}{2}$

when $x = \frac{l}{2}$ $y = 0$.

Hence the straight line *CDE*.

The B.M. at $\alpha\beta = Ax - \frac{W}{l}x \times \frac{x}{2} = Ax - \frac{W}{2l}x^2$.

Let y represent B.M.

Then $y = Ax - \frac{W}{2l}x^2 = \frac{W}{2}x - \frac{W}{2l}x^2$, *i.e.*, a parabola.

When $x = 0$ $y = 0$

When $x = \frac{l}{2}$ $y = \frac{Wl}{8}$

When $x = l$ $y = 0$.

Hence the curve *FGH*

The V.S.F. at any plane is the resultant of the forces to the left of the plane and the B.M. is the moment of that resultant about a point in the plane.

Let the adjoining Fig. 85 represent a simple horizontal beam supporting a load of W lbs. uniformly distributed over the first half of its length and a load of W lbs. concentrated at a point three-quarters of length from the first abutment A.

When x is not $> \frac{l}{2}$

$$\text{V.S.F.} = +A - \frac{2W}{l}x = W - \frac{2W}{l}x$$

When $x = 0$ V.S.F. $= +W$

When $x = \frac{l}{2}$ V.S.F. $= 0$

When x is $> \frac{l}{2}$ and $< \frac{3}{4}l$

$$\text{V.S.F.} = +A - W = 0$$

When x is $> \frac{3}{4}l$

$$\text{V.S.F.} = +A - W - W = -W$$

Hence the V.S.F. is represented by the ordinates to the line *CDEFG*

When x is not $> \frac{l}{2}$

$$\text{B.M.} = +Ax - \frac{2Wx}{l} \cdot \frac{x}{2} = Wx - \frac{W}{l}x^2 \text{ a parabola.}$$

When $x = 0$ B.M. $= 0$

When $x = \frac{l}{2}$ B.M. $= \frac{Wl}{4}$

When x is $> \frac{l}{2}$ and $< \frac{3}{4}l$

$$\text{B.M.} = Ax - W\left(x - \frac{l}{4}\right) = Wx - Wx + \frac{Wl}{4} = +\frac{Wl}{4}$$

i.e., the B.M. is constant as x varies between the limits

$$x = \frac{l}{2} \text{ and } x = \tfrac{3}{4}l$$

When x is $> \frac{3}{4}l$

$$\text{B.M.} = Ax - W\left(x - \frac{l}{4}\right) - W(x - \tfrac{3}{4}l)$$

$$= Wx - Wx + \frac{Wl}{4} - Wx + \tfrac{3}{4}Wl = Wl - Wx$$

A straight line

When $x = \frac{3}{4}l$ B.M. $= \dfrac{Wl}{4}$

When $x = l$ B.M. $= 0$

Hence the B.M. at any plane $\alpha\beta$ is represented by the ordinate to the line $HJKL$.

Questions

Draw the V.S.F. and B.M. diagrams for a simple horizontal beam supporting two loads of W lbs. each situated on points $\frac{1}{4}$ and $\frac{3}{4}$ of the span from the left hand abutment respectively.

Draw the V.S.F. and B.M. diagrams for a simple horizontal beam supporting a load of W lbs. distributed uniformly over $\frac{1}{2}$ of its length commencing at a point $\frac{1}{4}$ of length from the left hand abutment.

Let the adjoining Fig. 86 represent a simple horizontal cantilever supporting a load of W lbs. at its outer end.

Fig. 86.

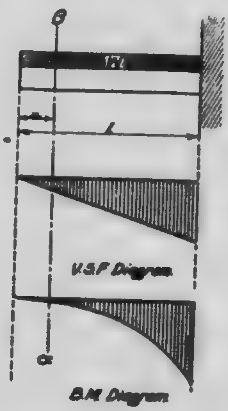


Fig. 87.

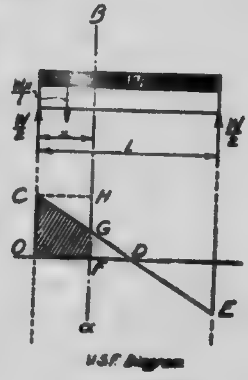


Fig. 88

The V.S.F. at $\alpha\beta = -W$ a straight line parallel to the axis of x. Hence V.S.F. diagram

The B.M. $= -Wx$

A straight line passing through the origin and when $x = l$

$$\text{B.M.} = -Wl$$

Hence the B.M. Diagram.

Let the adjoining Fig. 87 represent a simple horizontal cantilever supporting a load of W lbs. uniformly distributed over its length.

$$\text{V.S.F.} = -\frac{W}{l}x$$

$$\text{B.M.} = -\frac{W}{l}x \cdot \frac{x}{2} = -\frac{W}{2l}x^2$$

The area of the vertical shearing force diagram represents the bending moment.

Let the diagram, Fig. 88, represent a beam supporting a uniformly distributed load ... The vertical shearing force at any plane $\alpha\beta$ is the algebraic sum of the forces to the left of the plane, *i.e.*:

$$\text{V.S.F.} = +\frac{W}{2} - \frac{W}{l}x$$

When $x = 0$, V.S.F. $= \frac{W}{2}$, and V.S.F. $= 0$ when $x = \frac{l}{2}$; hence the ordinate to the straight line CDE at any plane $\alpha\beta$ represents the V.S.F. at that plane. Now the bending moment at the plane $\alpha\beta$ is the algebraic sum of the moments of the forces to the left of the plane about any point in the plane, *i.e.*: B.M. $= +\frac{W}{2} \cdot x - \frac{W}{l}x \cdot \frac{x}{2}$.

But in the V.S.F. diagram OC and CH represent $\frac{W}{2}$ and x respectively, therefore $\frac{W}{2}x$ may be represented to the area of the rectangle $OCHF$.

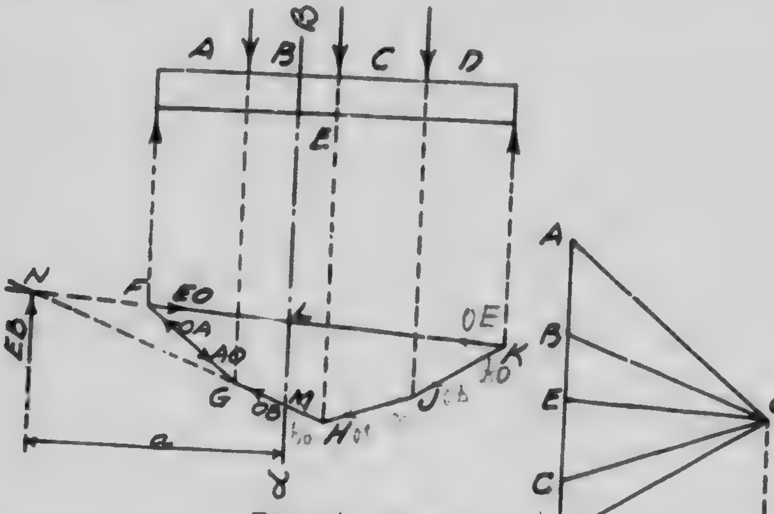
Again because FG represents the V.S.F. at $\alpha\beta$ it represents the difference between $\frac{W}{2}$ and $\frac{W}{l}x$ and as FH represents $\frac{W}{2}$ therefore HG represents $\frac{W}{l}x$.

Hence $\frac{W}{l}x \cdot \frac{x}{2}$ may be represented by $HG \cdot \frac{CH}{2}$, *i.e.*, by the area of the triangle CHG.

But B.M. $= +\frac{W}{2}x - \frac{W}{l}x \cdot \frac{x}{2}$ and may be represented by area of the rectangle OH less the area of the triangle CHG, or the area of the figure $OCGF$.

Hence the bending moment at any plane $\alpha\beta$ is represented by the area of the V.S.F. diagram to the left of the plane.

The V.S.F. is the resultant of the forces acting on the left of the plane and the B.M. is the moment of that resultant about a point in the plane.



Funicular Polygon

Fig. 89

Vector Polygon

Let Fig. 89 represent a [illegible] supporting three known loads AB, BC and CD as indicated. [illegible] the Vector Polygon $ABCD$. This polygon must close (as there is equilibrium) by the lines DE and EA where the point E is at present unknown. Select any point O and join it with the points A, B, C and D of the Vector Polygon. At any point G in the line of direction of the force AB replace it by its components AO and OB and produce the directions of these until they intersect the directions of EA and BC in F and H.

At the point F replace EA by its components EO and OA (the direction and magnitude of EO being unknown).

	At H replace	BC	by components	BO and OC
	At J "	CD "	"	CO and OD
and	at K "	DE "	"	DO and OE

(OE being unknown).

Thus the original five forces acting on the beam have been replaced by ten, and of these OA and AO act with equal magnitude and opposite senses in the same straight line. OA and AO are therefore in equilibrium.

 См. через страницу.

Moving Load

Let the diagram below, Fig. 90, represent a beam over which a load of W lbs. is to pass.

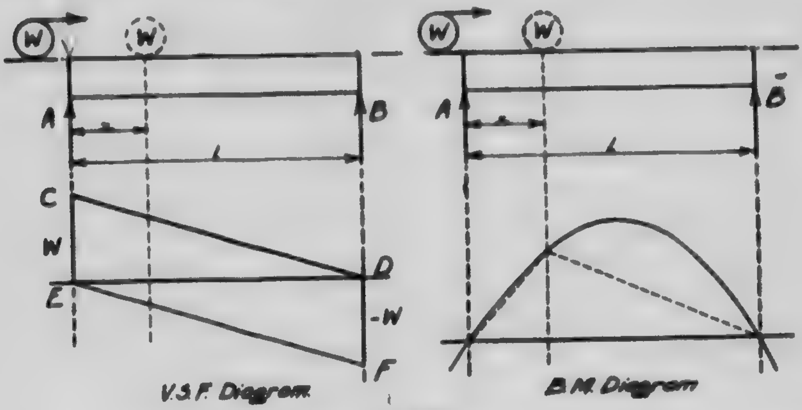


Fig. 90

When the load is a distance x from the left abutment A the V.S.F. at every point to the left of the load will be $+A$ and

$$A = \frac{x-l}{l} \text{ of } W.$$

To the right of the load V.S.F. $= -\frac{x}{l} W$.

Now as the load moves the $+ve$ value of the V.S.F. behind it changes and the negative value in front also changes as follows:

V.S.F. $= +\frac{x-l}{l} W$ and $-\frac{x}{l} W$

when $x=0$ V.S.F. $= +W$ and -0

when $x=l$ V.S.F. $= +0$ and $-W$.

Thus as the load passes A the V.S.F. behind it is $+W$ and as it continues to move this value changes according to the equation to a straight line and becomes 0 when it reaches the abutment B. Hence the line CD in Fig. 90. Similarly the ordinates to the line EF must represent the changing values of the V.S.F. in front of the load.

When the load is at a point distant x from A the B.M. for the beam is represented in the diagram by the dotted line, the maximum value being directly under the load. What is true for this position is true for every other position of the load.

$$\text{B.M. max.} = +Ax = +\frac{x-l}{l} Wx = \frac{W}{l}(lx - x^2)$$

Similarly OB and BO, OC and CO, and OD and DO are in equilibrium.

The forces EA, AB, BC; CD and DE are in equilibrium; therefore the forces

	EO and OA	acting at	F
	AO and OB	" "	G
	BO and OC	" "	H
	CO and OD	" "	J
and	DO and OE	" "	K

are in equilibrium.

Hence EO acting at F and OE acting at K are in equilibrium. Therefore, they must act in the same straight line KF.

Through O draw OE parallel to KF; then E is the point required to complete the Vector Polygon.

Let $\alpha\beta$ be any vertical plane intersecting the Funicular Polygon at L and M.

Consider the forces to the left of $\alpha\beta$, *i.e.*, EA and AB. These may be replaced by the component EO and OA acting at F and AO and OB at G. As OA and AO are equal in magnitude and opposite in sense, their resultant is 0. Then EA and AB are equivalent to EO acting at F and OB acting at G and their resultant must act through their intersection N. From the Vector Polygon it will be seen that the resultant of EO and OB is EB. Therefore the resultant of EA and AB is EB and acts at N.

The Bending Moment at $\alpha\beta$ is the algebraic sum of the moments of EA and AB about any point in the plane and is therefore equivalent to the moment of their resultant EB about a point in $\alpha\beta$.

Let the distance of EB from $\alpha\beta$ be a
then $\text{B.M.} = EB \times a$

The sides of the triangle NLM are parallel to those of the triangle OEB

$$\therefore \quad LM \times b = EB \times a$$
$$\text{But } EB \times a = \text{B.M.}$$
$$\therefore \quad \text{B.M.} = LM \times b$$

i.e., the ordinate of the funicular polygon at $\alpha\beta$ is proportional to the bending moment and may represent it.

To obtain from the figure the value of the bending moment measure the length a in inches and multiply by the scale of length used in drawing the diagram and measure EB in inches and multiply it by the scale of forces used in drawing the Vector Polygon. Thus

$$\begin{aligned} \text{B.M} &= EB \times a \times \text{scale of length} \times \text{scale of forces} \\ &= LM \times b \times \text{scale of length} \times \text{scale of forces} \\ &= LM \times (b \times \text{scale of length} \times \text{scale of forces}) \end{aligned}$$

hence the scale of B.M. for the Funicular Polygon is

b (measured in inches) $\times$ scale of length $\times$ scale of forces.

Thus the maximum value of the B.M. changes as the load moves from A to B from zero to zero according to the parabola

$$y=\frac{W}{l}(lx-x^2).$$

Draw the V.S.F. diagram (Fig. 91) for the moving load W as before and join CF, cutting the lines GJ and JK at the points H and K respectively.

The V.S.F. behind the load is $+A$.

The B.M. at x is $+Ax$ and may be represented by $GE \times GJ$, *i.e.*, by the area of the rectangle EJ.

But the triangle CGH = triangle KJH

Hence the area of the figure $ECHKL$ represents the B.M., *i.e.*, the area of the figure between the line CF and the axis of X to the left of the load represents the value of the maximum B.M. which occurs directly under the load.

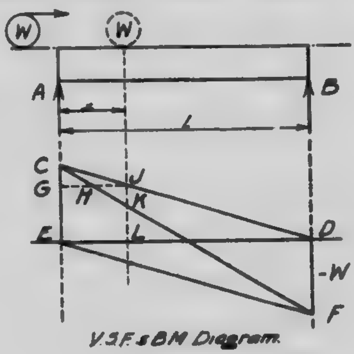


Fig. 91

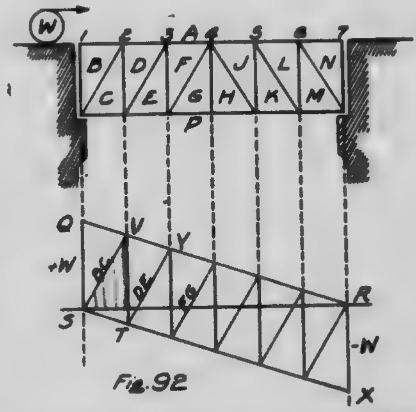


Fig. 92

Let Fig. 92 represent a Howe truss over which a load of W lbs. is to move. Consider the truss as a whole and draw the diagram of V.S.F., *i.e.*, QR and SX.

Next consider the member BC. The stress in this member depends on the amount of the shear it is called upon to resist and will therefore vary as the V.S.F. varies. When the load is at joint 1 the stress in BC is 0; when at 2 the vertical resolved part of the stress will be represented by the ordinate of the line QR directly below 2. As the load moves from joint 1 to 2 the member AB acts as a beam and places part of W on 1 and the remainder on 2, *i.e.*, when the load is 1/4 of the way, 1/4 of the W will be placed at 2, and when $1/n$ of the way, $1/n$th of W will act at 2. The part of W acting at 1 will not cause stress in any member of the truss other than the vertical member at the abutment.

Thus when the load is at 2 the vertical resolved part of the stress in BC is represented by the ordinate of QR directly below 2; hence

when W is $1n$th of the way from 1 to 2 the vertical resolved part of the stress in BC will be represented by $1/n$th of this ordinate. ∴ that part of the shear to be resisted by the member BC is represented by the ordinate to the straight lines SV and VR. Similarly the ordinates to the lines ST, TY, YR, represent the changes in the stress in DE.

It will be noticed that the line SVR lies entirely above the axis of X and hence the ordinate is always positive; therefore the member BC will always be in a state of compression. On the other hand part of the line $STYR$ lies below the axis and therefore the member DE will be in a state of tension while the load is moving from the point 1 to that point between 2 and 3 which lies directly above the point where the line TY cuts the axis, after which it will be in compression.

Therefore, the maximum tension will occur in DE when the load is at 2 and the maximum compression when at 3. Similarly the maximum tension in FG occurs when W is at 3 and the maximum compression when at 4. Thus the member BC will have to be designed to withstand more compression than either DE or FG while DE will be called on to withstand some tension and FG to resist twice this tension.

If in Fig. 92 Q and X were joined by a straight line the areas between this line and the axis of X would represent the changes in value of the B.M. as the load crosses the truss: therefore, the maximum tension will occur in the member CP when the load is at the point 2.

With the load at 2 imagine a plane through the area $ADCP$. Take moments about the point $ABCD$, considering the length of each panel 1 and its height h. Also consider moments with clockwise sense positive; then

$$\Sigma M = +PA1+0+0+0+CPh=0$$

$$\text{or } CP = -PA\frac{1}{h}$$

Hence the tension in the member CP.

Again take moments about $CDEP$

$$\Sigma M = +PA1+0+ADh+0+0=0$$

$$AD = -PA\frac{1}{h}$$

∴ the compression in the member AD is equal to the tension in CP.

Thus when W is at 2 the maximum tension occurs in CP and the maximum compression in AD.

Similarly when W is at 3, the maximum tension will occur in the member EP and the maximum compression in FA.

Again when W is at 4, the maximum tension will occur in the member GP.

The stress in the member CD will vary as that in DE and be equal to its vertical resolved part, *i.e.*, when DE is in compression CD must be in tension and *vice versa*.

Summary

When W is at point 2 the following maximum stresses occur:

BC —compression
CP —tension
CD —compression
DE —tension
DA —compression.

When W is at 3 the maximum stresses are compression DE, tension EP, compression EF, tension FG, compression FA.

When W is at 4 the maximum stresses are compression FG, tension GP.

The member GH will not be stresses as the load moves over the truss because ΣY must equal 0 for the point GHP at all times.

Questions

1. Determine the maximum stresses in the members of a Howe Truss of six panels when l is 6 ft. and h 10 ft. for a moving load of 12,000 lbs.

2. Determine the stresses in the same truss supporting dead loads of 1,000 lbs. at each joint along the upper chord.

3. Determine the maximum stresses in the same Howe Truss supporting both the live and dead loads given in questions 1 and 2.

4. If the diagonal members of the Howe truss considered in question 3 were designed to take compression only, where would *counter braces* become necessary and what compression would they be required to withstand?

Wind Pressure

$v=\sqrt{2gh}$ or $h=\frac{v^2}{2g}$

$p=h\times$wt. of a unit of volume

$p=\frac{v^2}{2g}\times$wt. of unit of volume

when the units are the ft. and sec.

The velocity of the wind is generally given in miles per hour.

Let V represent the velocity of wind in miles per hour

Then $v=\frac{V\times 5280}{60\times 60}$

Hence $p=\frac{V^2\times 5280^2}{60^4}\times$wt. of cubic ft. of air

The weight of 1 cubic foot of air at 60° and 760 mm. is .078.
Using as a maximum .1 $p = .0033 V^2$

Prof. C. F. Marvin (U.S.A. Signal Service) from experiments gives $p = .004 V^2$.

Mr. S. P. Langley from experiment found $p = .00315 V^2$.

Thus the pressure per sq. ft. (p) caused by the wind on a plane at right angles to its direction is probably somewhere between the above values, say $p = .0035 V^2$.

Thus a light wind of 10 miles per hour gives a pressure of .35 lbs. per sq. ft.

while a gale of 50 miles per hour
gives $p = 8.75$ lbs. per sq. ft.

and a hurricane of 80 miles per hour
gives $p = 22.4$ lbs. per sq. ft.

For a smooth plane inclined to the direction of the wind the pressure caused will be normal to the plane and probably given by $p' = p \sin a$ when a is the angle between the normal to the plane and the direction of the wind.

Let Fig. 93 represent a Fink truss hinged at the right wall, mounted on rollers at the left, and supporting known loads *DE*, *FG* and *GH* and at the same time resisting known wind pressures *BC*, *CD* and *EF*.

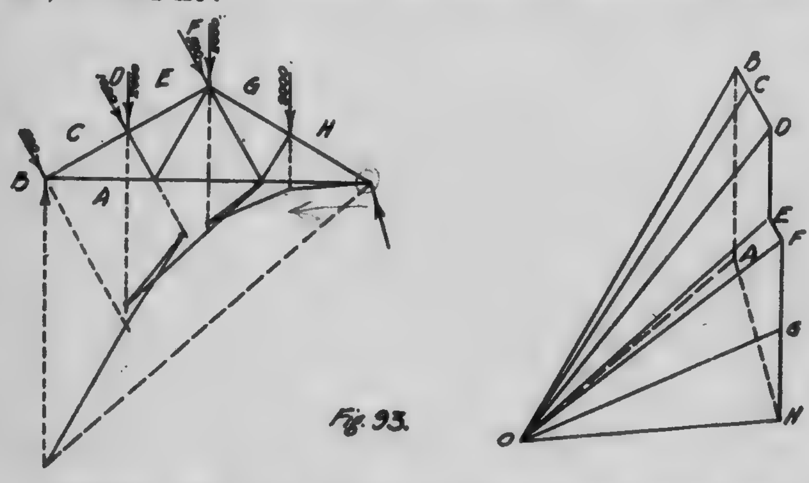


Fig. 93.

Construct the Vector Polygon except for the closing line *HA*, the direction of which is unknown.

Starting at the hinge *H* draw the Funicular Polygon and through *O* draw *OA* parallel to the closing line of the Funicular Polygon intersecting the line *AB* at *A*.

Join *HA*. *HA* then represents the right wall reaction in direction, magnitude and sense.

Inclined Plane

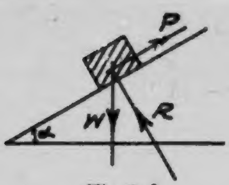


Fig. 94

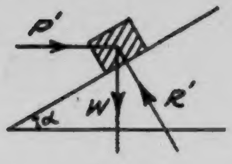


Fig. 95

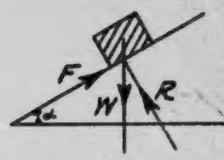


Fig. 96

Let Fig. 94 represent a body resting on a perfectly smooth plane inclined to the horizontal at an angle a and kept in equilibrium by a pull of P lbs.

The forces acting on the body are its weight W, the pull P, and the pressure of the plane R which must be normal to its surfaces.

These forces must be in equilibrium; hence $R = W \cos a$ and $P = W \sin a$ as they must be equal to the resolved parts of W in the directions of R and P.

Suppose the balancing force P' to be applied in a horizontal direction as indicated in Fig. 95. Then $R' = \dfrac{W}{\cos a}$ and $P' = W \tan a$

Question

Draw the Vector Polygons for the forces in Figs. 94, 95 and 96.

Friction

The sliding friction between two surfaces depends on the nature of the surfaces and on the pressure between them; but, is independent of the area.

The proof of this statement is experimental.

It has been determined that when a body rests on a horizontal plane a definite horizontal force must be applied to cause it to move. If a weight equal to that of the body is placed on it, the pull required to move it will have to be doubled.

Again, when the area exposed to friction is reduced it has been found that while the weight remains constant, the pull necessary to overcome friction remains constant.

The direction and sense of the friction will be opposite to that of the motion or the tendency towards motion. Thus, if a body is resting on an inclined plane, Fig. 96, the friction will act upward while if an attempt is made to move the body up grade friction will act downward.

Thus if a body rests in equilibrium on a plane inclined to the horizontal at an angle a the amount of friction F must equal $W \sin a$. Now if a is increased until the body is just on the point of

sliding down, α is called the limiting angle of friction or the angle of repose $F=W \sin \alpha$ $R=W \cos \alpha$ or $W=\dfrac{R}{\cos \alpha}$

$$\therefore \quad F=\frac{R}{\cos} \sin \alpha=R \tan \alpha$$

i.e., the friction between the surfaces is equal to the pressure between them multiplied by the fraction $\tan \alpha$. Such a fraction is called the coefficient of friction.

Thus the tangent of the angle of repose is equal to the co-efficient of friction.

The mechanical screw is an inclined plane wrapped around a cylinder and the inclination of the plane is given by the pitch of the screw.

Question

What pressure applied horizontally on the outer end of the wrench or handle of a screw jack will be necessary to lift a weight of one ton when the leverage of the handle is 5. The diameter of the screw is 2 inches with four threads per inch. The co-efficient of friction between the nut and the screw is 0.1.

Pulleys

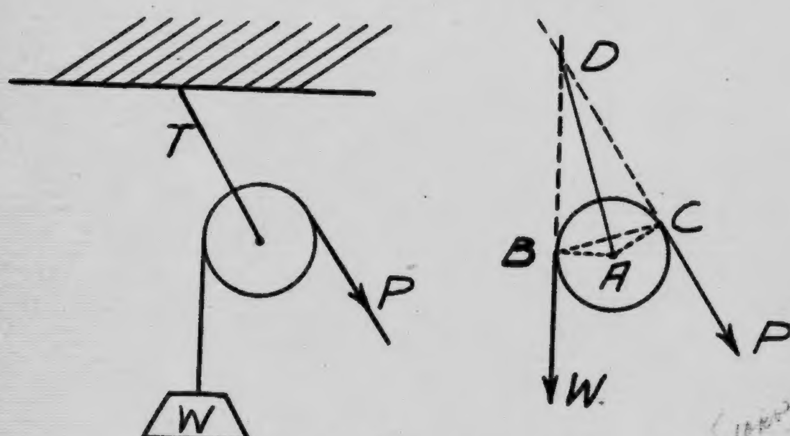


Fig 97

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With a single pulley used as in Fig. 97 it would assume some such position as that indicated. The sheave is acted on by three forces W vertically downwards, P in the direction of the rope and T the pull of the fastening. Suppose these forces to be in equilibrium.

The magnitude, direction and sense of W are known; the magnitude and direction of T are unknown, while the magnitude of P is also unknown.

Produce the lines of direction of P and W until they intersect at D.

The resultant of W and P must act through D; therefore the balancing force or T must act through D; but, it also acts through the point A; therefore AD is the direction of T.

Join BC. Then triangle ABC is isosceles; and therefore, triangle BDC is isosceles; hence DA bisects triangle BDC and the angle BDA = angle CDA.

Consider the resolved parts of P, W and T in the direction at right angles to AD. The resolved part of T in this direction is nothing; therefore the resolved parts of W and P must be equal; and as the angles BDA and CDA are equal, the force P must be equal to W.

Thus when a line passes over a pulley, the tension of the portion to the right of the sheave must be equal to that on the left.

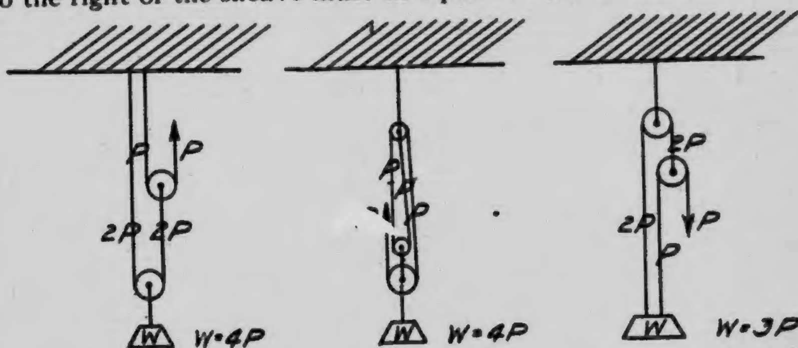

Fig. 98.

It is apparent that the mechanical efficiency in the above cases, Fig. 98, are 4, 4 and 3.

In the Weston differential pulley the double block at the top has the small and large sheaves cast as one and the chain passing over both cannot slip.

Consider first the single pulley at the bottom $W=2Q$.

Let r_1 be the radius of the large sheave of the upper pulley and r_2 the radius of the smaller.

Take moments about the centre of the pulley

$$\Sigma M=0$$

$$-Qr_1+0+Qr_2+P\cdot r_1=0$$

$$\therefore\ Pr_1=Q(r_1-r_2)$$

$$P=\frac{W}{2}\ \frac{(r_1-r_2)}{r_1}$$

The mechanical advantage is therefore $\frac{r_1 - r_2}{2r_1}$

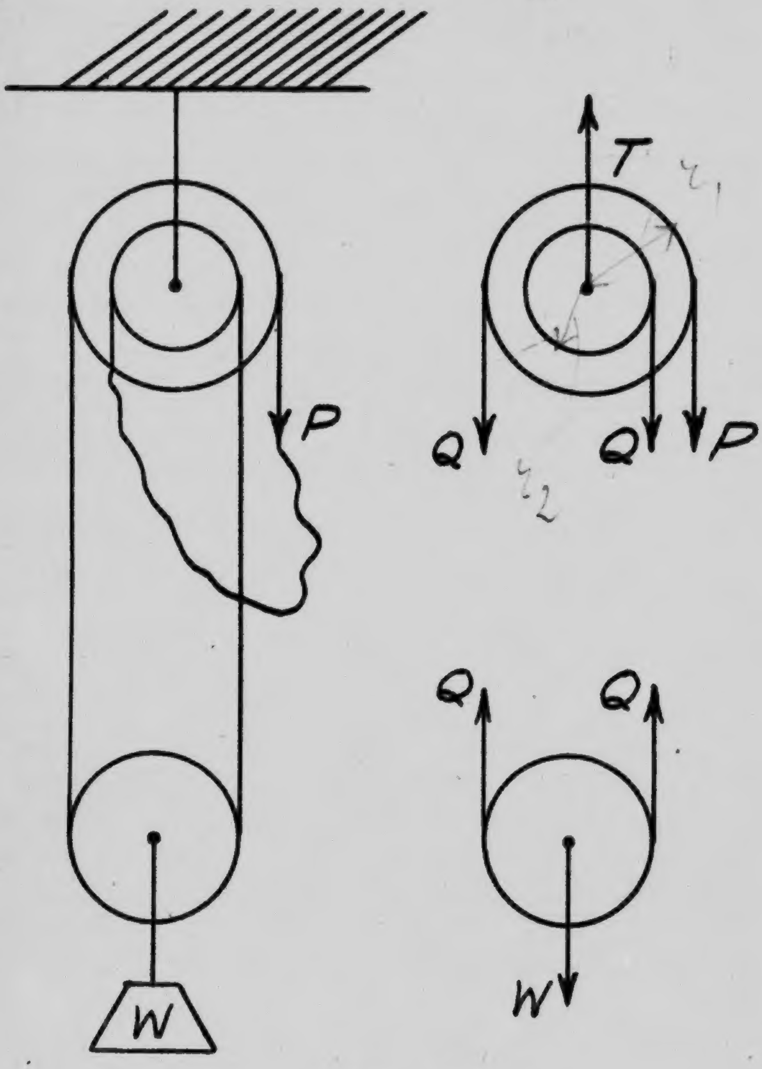

Fig. 99.

Hence the smaller the difference between r_1 and r_2, the greater the advantage and the slower the pulley will act.

This is sometimes called the differential chain block.